

Asymptotic Optimality of Massive MIMO Systems Using Densely Spaced Transmit Antennas

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Abstract

This paper considers a deterministic physical model of massive multiple-input multiple-output (MIMO) systems with uniform linear antenna arrays. It is known that the maximum spatial degrees of freedom is achieved by spacing antenna elements at half the carrier wavelength. The purpose of this paper is to investigate the impacts of spacing antennas more densely than the critical separation. The achievable rates of MIMO systems are evaluated in the large-system limit, where the lengths of transmit and receive antenna arrays tend to infinity with the antenna separations kept constant. The main results are twofold: One is that, under a mild assumption of channel instances, spacing antennas densely cannot improve the capacity of MIMO systems normalized by the spatial degrees of freedom. The other is that the normalized achievable rate of quadrature phase-shift keying converges to the normalized capacity achieved by optimal Gaussian signaling, as the transmit antenna separation tends to zero after taking the large-system limit. The latter result is based on mathematical similarity between MIMO transmission and faster-than-Nyquist signaling in signal space representations.

Index Terms

Massive multiple-input multiple-output (MIMO) systems, uniform linear antenna arrays, antenna spacing, faster-than-Nyquist signaling, large-system analysis.

I. INTRODUCTION

A. Motivation

MASSIVE multiple-input multiple-output (MIMO) systems [2]–[5] are promising schemes for future wireless communications. In massive MIMO systems, very large antenna arrays are used to attain many spatial degrees of freedom. It is an important topic in information theory to elucidate the benefit obtained by utilizing such antenna arrays.

Spatial correlations are a key factor that affects the performance of MIMO systems. In early work [6], Telatar proved that, when the channel matrices have independent and identically distributed (i.i.d.) circularly symmetric complex Gaussian (CSCG) elements, the channel capacity of MIMO systems is proportional to the minimum of the numbers of transmit and receive antennas. However, experimental channel measurements [7], [8] demonstrated that this idealized assumption is broken in realistic MIMO systems, while the Gaussianity of each channel gain may be satisfied. Thus, the influence of spatial correlations has to be taken into account to understand the potential of MIMO systems.

Spatial correlations are mainly caused by multipath fading and antenna properties, such as antenna spacing and mutual coupling between adjacent antenna elements [9]. Mutual coupling is a phenomenon that occurs between closely spaced antenna elements, and results in spatial correlations between transmit antennas and between receive antennas [10]. The influence of mutual coupling may be mitigated by constructing matching networks at both transmit and receive sides [11]–[13]. Thus, this paper considers idealized MIMO systems with no mutual coupling, and focuses on the impacts of multipath fading and antenna spacing.

For simplicity, consider uni-polarized uniform linear antenna arrays. Poon *et al.* [14], [15] proved that the spatial degrees of freedom are at most $2\min\{L_t, L_r\}$ in general, in which L_t and L_r represent the lengths of the transmit and receive antenna arrays normalized by the carrier wavelength, respectively. When the normalized separations Δ_t and Δ_r of transmit and receive antenna elements are equal to the critical value $1/2$, the early result $\min\{L_t/\Delta_t, L_r/\Delta_r\}$ by Telatar [6] is consistent with the general result. Thus, the spatial degrees of freedom are dominated by the normalized array lengths.

In order to explain the motivation of this paper, we shall review the signal-space approach in [14]. A continuous transmit antenna array is analogous to a band-limited system with bandwidth $W = 1$. The spatial domain $[-L_t/2, L_t/2]$ corresponds to the time domain, whereas the angular domain $[-1, 1]$ is associated with the frequency domain $[-W, W]$. The classical sampling theorem implies that, when L_t tends to infinity, any *continuous-time* signal in the spatial domain can be re-constructed by sampling the signal at antenna separation $\Delta_t = 1/(2W)$ corresponding to the Nyquist period. Thus, there are no points in spacing antenna elements more densely than the critically spaced case $\Delta_t = 1/2$, as long as continuous Gaussian signaling is used.

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The purpose of this paper is to investigate the influence of the transmit antenna separation for the case of suboptimal digital modulation. It has been shown that the performance can be improved by using a symbol period shorter than the critical Nyquist period, called faster-than-Nyquist (FTN) signaling [16]–[18], when quadrature phase-shift keying (QPSK) is used. More precisely, Yoo and Cho [18] proved that QPSK FTN signaling¹ can achieve the channel capacity when the symbol period tends to zero. This result motivates us to investigate the densely spaced case $\Delta_t < 1/2$ for QPSK.

B. Methodology

A well-established methodology for massive MIMO systems is the large-system analysis, in which the numbers of transmit and receive antennas are assumed to tend toward infinity at the same rate. In conventional large-system analysis, random matrix theory [20]–[25] or the replica method [26]–[33] developed in statistical physics was utilized to analyze the performance of MIMO systems under the assumption of statistical channel models. The large-system analysis has been accepted from two points of view. One is that the convergence of the large-system limit is so quick that asymptotic results can provide good approximations for finite-sized systems [34]. The other is that technological innovation [4], [5] is increasing the number of antennas to be regarded as a realistic number, while there is still a limitation in the number of radio-frequency (RF) chains that can be equipped in massive MIMO transmitters. See e.g. [35]–[38] for a solution to the RF chain issue.

We shall present a brief history of existing large-system analyses. It should be noted that it is of course impossible to review all efforts. In early works, channel matrices with i.i.d. zero-mean elements were investigated for Gaussian [20], [21] and non-Gaussian [26], [28] signaling. Spatially correlated channel matrices were analyzed in [22], [27] for Gaussian signaling. After the characterization of capacity-achieving input distributions for the spatially correlated case [39]–[41], the same methodology was applied to the precoder-design issue for Gaussian [23], [24] and non-Gaussian [29], [30], [33] signaling, as well as to the channel-estimation issue [25], [31], [32].

In order to investigate the influence of the transmit antenna separation, this paper considers a deterministic physical model of MIMO systems with uniform linear antenna arrays [42], rather than statistical channel models. All of the above existing works are based on statistical channel models. The conventional large-system limit corresponds to the limit in which the normalized lengths of the transmit and receive antenna arrays tend to infinity at the same rate with the normalized antenna separations kept constant. Thus, the latter limit—called large-system limit—is considered in this paper.

The methodology presented in this paper is based on the notion of asymptotically equivalent matrices in the large-system limit. The notion provides an elementary proof of the classical Szegő theorem for Toeplitz matrices [43]. We compare the performance of massive MIMO systems with different antenna separations, by applying the same methodology to the corresponding channel matrices.

C. Contributions

The main contributions of this paper are twofold.

- We evaluate the achievable rate of massive MIMO systems with precoded Gaussian signaling—referred to as constrained capacity—normalized by the maximum spatial degrees of freedom. It is proved that there are no points in considering densely spaced antenna elements in terms of the normalized constrained capacity, with the exception of power gain. More precisely, the normalized constrained capacity for the densely spaced case is asymptotically equal to that for the critically spaced case under an appropriate scaling of signal-to-noise ratio (SNR).

This statement itself is intuitively expected and proved in the high SNR regime [15]. However, to the best of author's knowledge, no rigorous proof is known for finite SNRs.

- For non-Gaussian signaling, we consider the limit in which the transmit antenna separation tends to zero after taking the large-system limit. It is proved that the normalized achievable rate for precoded QPSK signaling converges to the corresponding constrained capacity achieved by precoded Gaussian signaling. This result is analogous to the asymptotic optimality of QPSK FTN signaling [18].

The result implies that the transmission scheme with bounded peak power can approximately achieve the constrained capacity for all SNRs, by using transmit antenna arrays with densely spaced antenna elements, whereas the peak power is unbounded for Gaussian signaling. Note that, due to precoding, the peak power tends to infinity in order to achieve the constrained capacity exactly. Thus, this statement does not contradict existing results with respect to peak power constraints [44]–[46].

¹ Note that discrete FTN signaling does not contradict the sampling Theorem, which claims that there is a one-to-one correspondence between any continuous-time real-valued signal and the associated *real-valued* samples at the Nyquist rate. Discrete-valued samples do not necessarily represent all continuous-time real-valued signals.

Instead of sending signals drawn from a memoryless Gaussian source, let us consider transmission of the corresponding quantized symbols on a finite alphabet. The rate-distortion theory [19] implies that a diverging compression rate is required to achieve arbitrarily small mean-square-error distortion. This implies that a vanishing symbol period is needed in order for this scheme to achieve the transmission rate of Gaussian signaling for any signal-to-noise ratio.

D. Organization

The remainder of this paper is organized as follows: After presenting the notation used in this paper, in Section II the basic properties of uniform linear antenna arrays are reviewed. Technical key lemmas for proving the main results are also derived, while the proofs are given in Appendices A and B.

In Section III a deterministic physical model of MIMO systems is introduced. Furthermore, the angular domain representation of the channel model is reviewed. The representation is useful for proving the main results.

Section IV presents three main theorems. One of the main theorems is proved in Section V. The proofs of the other two theorems are given in Appendices C and D on the basis of the notion of asymptotically equivalent matrices. After presenting numerical results in Section VI, this paper is concluded in Section VII.

E. Notation

As basic parameters, we use the normalized length L_t of transmit antenna arrays, the normalized transmit antenna separation Δ_t , the number $M = L_t/\Delta_t$ of transmit antennas, the normalized length L_r of receive antenna arrays, the normalized receive antenna separation Δ_r , and the number $N = L_r/\Delta_r$ of receive antennas. The load $\alpha = L_t/L_r$ is defined as the ratio of the transmit array length to the receive array length, while *spurious* load $\beta = M/N$ is given by the ratio of the number of transmit antennas to the number of receive antennas.

For integers $a, b \in \mathbb{Z}$ satisfying $a < b$, the set $\{a, a+1, \dots, b-1\}$ of consecutive integers is simply represented as $[a : b)$, while $[a, b]$ denotes the interval for real $a, b \in \mathbb{R}$. We define the sets $[a : b]$, $(a : b]$, and $(a : b)$ in the same manner. For any sequence $\{a_n\}$, the convention $\sum_{n \in \mathcal{N}} a_n = 0$ is introduced when the set \mathcal{N} of indices is empty. In this paper, the indices of vectors and matrices start from 0.

The imaginary unit is denoted by j . For a complex number $z \in \mathbb{C}$ and a complex matrix \mathbf{A} , the notations z^* , \mathbf{A}^T , and \mathbf{A}^H represent the complex conjugate of z , the transpose of \mathbf{A} , and the conjugate transpose of \mathbf{A} , respectively. The matrix $\mathbf{E}_{n,k}(\mathbf{A})$ is defined as the extended matrix obtained by inserting k all-zero columns and k all-zero rows after the first n columns and rows of \mathbf{A} , respectively, i.e.

$$\mathbf{E}_{n,k}(\mathbf{A}) = \begin{pmatrix} \mathbf{A}_{00} & \mathbf{O}_{n \times k} & \mathbf{A}_{01} \\ \mathbf{O}_{k \times n} & \mathbf{O}_{k \times k} & \mathbf{O} \\ \mathbf{A}_{10} & \mathbf{O} & \mathbf{A}_{11} \end{pmatrix}, \quad (1)$$

with

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{00} & \mathbf{A}_{01} \\ \mathbf{A}_{10} & \mathbf{A}_{11} \end{pmatrix}, \quad \mathbf{A}_{00} \in \mathbb{C}^{n \times n}. \quad (2)$$

Let $\delta_{n,m}$ denote the Kronecker delta. The standard orthonormal basis of \mathbb{R}^N is written as $\{\mathbf{e}_{N,n} \in \mathbb{R}^N | n \in [0 : N)\}$, i.e. $(\mathbf{e}_{N,n})_{n'} = \delta_{n,n'}$. For a complex vector \mathbf{v} , the Euclidean norm is written as $\|\mathbf{v}\|$.

The notation $\mathbf{a} \sim \mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ indicates that the random vector \mathbf{a} follows the proper complex Gaussian distribution with mean $\mathbb{E}[\mathbf{v}] = \boldsymbol{\mu}$ and covariance $\mathbb{E}[\mathbf{a}\mathbf{a}^H] = \boldsymbol{\Sigma}$. We use the standard notation [19] for information-theoretical quantities such as mutual information.

II. UNIFORM LINEAR ANTENNA ARRAY

A. Basic Properties

We review properties of uniform linear antenna arrays [42, Chapter 7] with no mutual coupling. Let L and Δ denote the antenna length and the antenna separation normalized by the carrier wavelength, respectively. The number of antenna elements is given by $N = L/\Delta$, which must be a positive integer. Without loss of generality, we assume that L is an integer. If L is not, L is replaced by $\lfloor L \rfloor$. This is equivalent to assuming a rational antenna separation $\Delta = L/N \in \mathbb{Q}$.

Let $\mathbf{s}_{L,\Delta}(\Omega) \in \mathbb{C}^N$ denote the unit spatial signature with respect to directional cosine $\Omega = \cos \phi$ for $\phi \in [0, 2\pi)$

$$\mathbf{s}_{L,\Delta}(\Omega) = \frac{1}{\sqrt{L/\Delta}} \left(1, e^{-2\pi j \Delta \Omega}, \dots, e^{-2\pi j (L/\Delta - 1) \Delta \Omega} \right)^T. \quad (3)$$

The standard inner product between two vectors $\mathbf{s}_{L,\Delta}(\Omega)$ and $\mathbf{s}_{L,\Delta}(\Omega')$ depends on Ω and Ω' only through the difference $\Omega - \Omega' \in [-2, 2]$, and is given by

$$\begin{aligned} f_{L,\Delta}(\Omega - \Omega') &= \mathbf{s}_{L,\Delta}(\Omega')^H \mathbf{s}_{L,\Delta}(\Omega) \\ &= \frac{\Delta}{L} \sum_{n=0}^{L/\Delta - 1} e^{-2\pi j n \Delta (\Omega - \Omega')}. \end{aligned} \quad (4)$$

Properties of the antenna array are characterized by the beamforming pattern $(\phi, |f_{L,\Delta}(\cos \phi_0 - \cos \phi)|)$ with respect to direction $\phi_0 \in [0, 2\pi)$.

In order to investigate properties of the beamforming pattern, we extend the domain of $f_{L,\Delta}(\Omega)$ from $[-2, 2]$ to \mathbb{R} . The function $f_{L,\Delta}(\Omega)$ has the following properties [42]:

Property 1:

- $f_{L,\Delta}(\Omega)$ is a periodic function with period $1/\Delta$.
- $f_{L,\Delta}(-\Omega) = f_{L,\Delta}^*(\Omega)$. Thus, $|f_{L,\Delta}(\Omega)|$ is even.
- For all integers $k \in \mathbb{Z}$, $f_{L,\Delta}(k/L) = 1$ if k is a multiple of L/Δ . Otherwise, $f_{L,\Delta}(k/L) = 0$.
- $|f_{L,\Delta}(\Omega)| \leq 1$.
-

$$f_{L,\Delta}(\Omega) = e^{\pi j(L-\Delta)\Omega} \text{sinc}_{L/\Delta}(L\Omega), \quad (5)$$

with

$$\text{sinc}_N(x) = \frac{1}{N} \frac{\sin(\pi x)}{\sin(\pi x/N)}, \quad (6)$$

where we define $\text{sinc}_N(x) = 1$ at $x = kN$ for all $k \in \mathbb{Z}$.

Note that the behavior of (6) is similar to that of the sinc function $\text{sinc}(x) = \sin(\pi x)/\pi x$. In fact, we have the point-wise convergence $\lim_{N \rightarrow \infty} \text{sinc}_N(x) = \text{sinc}(x)$. Furthermore, the following upper bound holds:

$$|\text{sinc}_N(x)| \leq \frac{1}{2x}, \quad \text{for all } x \in (0, N/2], \quad (7)$$

since

$$x|\text{sinc}_N(x)| \leq \frac{1}{\pi} \frac{\pi x/N}{\sin(\pi x/N)} \leq \frac{1}{2}, \quad (8)$$

where the last inequality is due to the fact that $u/\sin(u)$ is monotonically increasing for $u \in (0, \pi/2]$.

B. Basis Expansion

We next introduce a basis expansion of the spatial signature. The third property in Property 1 implies that the signature vectors $\{\mathbf{s}_{L,\Delta}(k/L) | k \in [0 : L/\Delta]\}$ form an orthonormal basis of \mathbb{C}^N . Representing the spatial signature $\mathbf{s}_{L,\Delta}(\Omega)$ with the basis yields

$$\mathbf{s}_{L,\Delta}(\Omega) = \sum_{k=0}^{L/\Delta-1} f_{L,\Delta}\left(\frac{k}{L} - \Omega\right) \mathbf{s}_{L,\Delta}\left(\frac{k}{L}\right). \quad (9)$$

Figure 1 shows the properties of $f_{L,\Delta}(k/L - \Omega)$ for the critically spaced case $\Delta = 1/2$ and the densely spaced case $\Delta < 1/2$. There is approximate correspondence between $f_{L,1/2}(k/L - \Omega)$ and $f_{L,\Delta}(k/L - \Omega)$ with $\Delta < 1/2$ for $k \in [0 : L)$, and between $f_{L,1/2}(k/L - \Omega)$ and $f_{L,\Delta}((k + L/\Delta - 2L)/L - \Omega)$ for $k \in (L : 2L)$. Furthermore, the beamforming pattern for $k = L$ and $\Delta = 1/2$ approximately coincides with a superposition of those for $k = L$ and $k = L/\Delta - L$ in the densely spaced case $\Delta < 1/2$. The beamforming patterns for the densely spaced case have no main lobes when $k \in (L : L/\Delta - L)$.

We shall present three technical key lemmas to justify these observations. A first lemma implies that channel gains introduced in the next section are adequately normalized.

Lemma 1: For $\Omega, \Omega' \in [-1, 1]$, define

$$I_1(\Omega, \Omega'; L, \Delta) = \sum_{k=0}^{L/\Delta-1} f_{L,\Delta}\left(\frac{k}{L} - \Omega\right) f_{L,\Delta}^*\left(\frac{k}{L} - \Omega'\right). \quad (10)$$

Then,

$$I_1(\Omega, \Omega'; L, \Delta) = f_{L,\Delta}(\Omega' - \Omega). \quad (11)$$

In particular, $|I_1(\Omega, \Omega'; L, \Delta)| \leq 1$ holds.

Proof: Note $N = L/\Delta$. Substituting (4) into (10) yields

$$\begin{aligned} I_1 &= \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{n'=0}^{N-1} e^{2\pi j \Delta (n\Omega - n'\Omega')} \sum_{k=0}^{N-1} e^{-2\pi j (n-n')k/N} \\ &= \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi j n \Delta (\Omega - \Omega')}, \end{aligned} \quad (12)$$

which implies (11) from (4). The last statement in the lemma holds from the fourth property in Property 1. ■

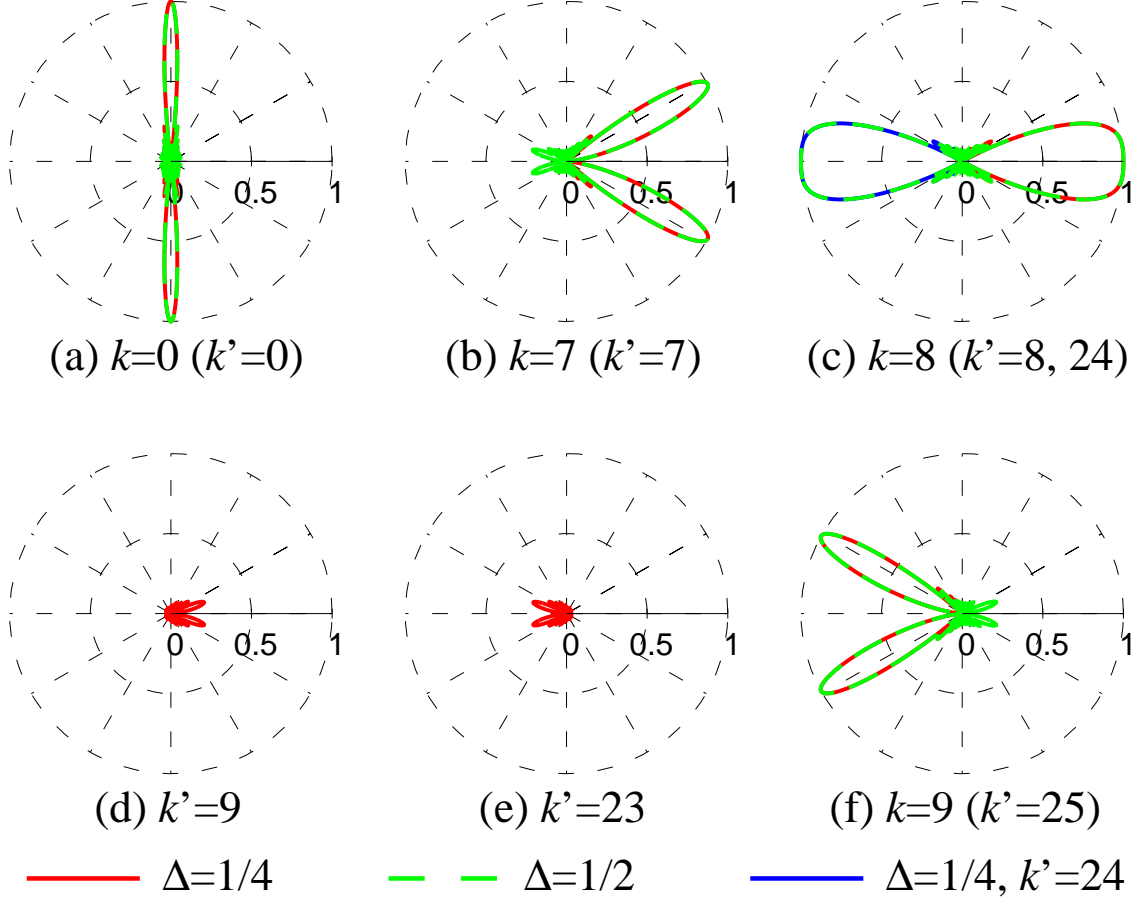


Fig. 1. Beamforming patterns $(\phi, |f_{L,\Delta}(k/L - \cos \phi)|)$ for the critically spaced case $\Delta = 1/2$, and $(\phi, |f_{L,\Delta}(k'/L - \cos \phi)|)$ for the densely spaced case $\Delta = 1/4$. The normalized length of the antenna array is set to $L = 8$.

In order to present the remaining two lemmas, we decompose the set $[0 : L/\Delta]$ of antenna indices into the disjoint subsets $\{L\} \cup \mathcal{K}_1 \cup \mathcal{K}_2$, with

$$\begin{aligned} \mathcal{K}_1 &= [0 : L) \cup [L/\Delta - L : L/\Delta), \\ \mathcal{K}_2 &= (L : L/\Delta - L). \end{aligned} \quad (13)$$

For $\Delta = 1/2$, we define $\mathcal{K}_1 = [0 : L/\Delta]$ and $\mathcal{K}_2 = \emptyset$.

For $l \in \mathbb{N}$, let $\{s_l \in \mathbb{N}\}$ denote a sequence of positive integers that satisfies two conditions: $\lim_{l \rightarrow \infty} s_l = \infty$ and $\lim_{l \rightarrow \infty} s_l/l = 0$. We restrict the domain of the directional cosine Ω from $[-1, 1]$ to $\mathcal{D}_L = [-(1 - s_L/L), 1 - s_L/L]$. The neighborhoods of the boundaries in the angular domain $[-1, 1]$ contribute to neighborhoods of the two indices L and $L/\Delta - L$ in the spatial domain. Since the contribution is small but non-negligible, we consider the restricted interval \mathcal{D}_L in the angular domain.

Lemma 2: For $\Omega, \Omega' \in \mathcal{D}_L$, define

$$I_2(\Omega, \Omega'; L, \Delta) = \sum_{k \in \mathcal{K}_2} f_{L,\Delta} \left(\frac{k}{L} - \Omega \right) f_{L,\Delta}^* \left(\frac{k}{L} - \Omega' \right). \quad (14)$$

Then, there exists some constant $A > 0$ such that

$$s_L |I_2(\Omega, \Omega'; L, \Delta)| < A, \quad (15)$$

for all $\Omega, \Omega' \in \mathcal{D}_L$ and all $\Delta \in (0, 1/2]$ in the limit $L, N \rightarrow \infty$ with $\Delta = L/N$ fixed.

Proof: See Appendix A. ■

Lemma 3: Let the one-to-one mapping $\kappa_{L,\Delta}(k)$ from $[0 : 2L]$ onto $\mathcal{K}_1 \cup \{L\}$ as

$$\kappa_{L,\Delta}(k) = \begin{cases} k & \text{for } k \in [0 : L), \\ k + L/\Delta - 2L & \text{for } k \in [L : 2L), \\ L & \text{for } k = 2L. \end{cases} \quad (16)$$

For $\Omega, \Omega' \in \mathcal{D}_L$, define

$$I_3(\Omega, \Omega'; L, \Delta) = \sum_{k=0}^{2L-1} D_{k,\kappa_{L,\Delta}(k)}(\Omega) D_{k,\kappa_{L,\Delta}(k)}^*(\Omega') + f_{L,\Delta}(1 - \Omega) f_{L,\Delta}^*(1 - \Omega'), \quad (17)$$

with

$$D_{k,k'}(\Omega) = f_{L,1/2} \left(\frac{k}{L} - \Omega \right) - f_{L,\Delta} \left(\frac{k'}{L} - \Omega \right). \quad (18)$$

Then, there exists some constant $A > 0$ such that

$$s_L |I_3(\Omega, \Omega'; L, \Delta)| < A, \quad (19)$$

for all $\Omega, \Omega' \in \mathcal{D}_L$ and all $\Delta \in (0, 1/2]$ in the limit $L, N \rightarrow \infty$ with $\Delta = L/N$ fixed.

Proof: See Appendix B. ■

All lemmas depend highly on the properties of the function $f_{L,\Delta}(\Omega)$. Lemma 2 indicates that most of the energy is concentrated on the antenna indices $\mathcal{K}_1 \cup \{L\}$. Lemma 3 will be used to evaluate the capacity difference between the critically spaced case and the densely spaced case.

III. CHANNEL MODEL

A. MIMO Channel

Consider MIMO channels with M transmit antennas and N receive antennas. The received vector $\mathbf{y} \in \mathbb{C}^N$ is given by

$$\mathbf{y} = \sqrt{\gamma} \mathbf{H} \mathbf{x} + \mathbf{w}. \quad (20)$$

In (20), $\mathbf{H} \in \mathbb{C}^{N \times M}$, $\mathbf{x} \in \mathbb{C}^M$, and $\mathbf{w} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_N)$ denote the channel matrix, the transmitted vector, and the additive white Gaussian noise (AWGN) vector, respectively. The average power constraint $\mathbb{E}[\|\mathbf{x}\|^2] \leq 1$ is imposed. The parameter $\gamma > 0$ corresponds to the SNR.

B. Physical Modeling

Uniform linear antenna arrays with no mutual coupling are assumed. Let Δ_t and Δ_r denote the transmit and receive antenna separation normalized by the carrier wavelength, respectively. The normalized lengths of the transmit and receive antenna arrays are given by $L_t = M\Delta_t$ and $L_r = N\Delta_r$. A deterministic physical model [42, Eq. (7.56)] of the channel matrix $\mathbf{H} \in \mathbb{C}^{N \times M}$ is given by

$$\mathbf{H} = \sqrt{NM} \int a(p) \mathbf{s}_{L_r, \Delta_r}(\Omega_r(p)) \mathbf{s}_{L_t, \Delta_t}(\Omega_t(p))^H dp, \quad (21)$$

where the transmit and receive unit spatial signature vectors $\mathbf{s}_{L_t, \Delta_t}(\Omega) \in \mathbb{C}^M$ and $\mathbf{s}_{L_r, \Delta_r}(\Omega) \in \mathbb{C}^N$ with respect to directional cosine Ω are defined by (3). In (21), $a(p) \in \mathbb{C}$ represents the complex attenuation of path p . The directional cosines $\Omega_t(p) = \cos \phi_t(p) \in [-1, 1]$ and $\Omega_r(p) = \cos \phi_r(p) \in [-1, 1]$ are defined via the departure angle $\phi_t(p)$ from the transmit antenna array and via the incidence angle $\phi_r(p)$ to the receive antenna array for path p .

C. Angular Domain Representation

We next introduce the angular domain representation of the channel matrix \mathbf{H} . Substituting the basis expansions (9) for $\mathbf{s}_{L_t, \Delta_t}(\Omega_t(p))$ and $\mathbf{s}_{L_r, \Delta_r}(\Omega_r(p))$ into (21), we find that the channel matrix (21) can be represented as

$$\sqrt{\gamma} \mathbf{H} = \sqrt{\tilde{\gamma}} \mathbf{U}_{L_r, \Delta_r} \mathbf{G}_{\Delta_t, \Delta_r} \mathbf{U}_{L_t, \Delta_t}^H, \quad (22)$$

with

$$\tilde{\gamma} = \frac{\gamma}{4\Delta_t \Delta_r}. \quad (23)$$

In (22), the $N \times N$ unitary matrix $\mathbf{U}_{L_r, \Delta_r}$ has $\mathbf{s}_{L_r, \Delta_r}(n/L_r)$ as the n th column for $n = 0, \dots, N-1$, while the $M \times M$ unitary matrix $\mathbf{U}_{L_t, \Delta_t}$ has $\mathbf{s}_{L_t, \Delta_t}(m/L_t)$ as the m th column for $m = 0, \dots, M-1$. Note that the $\mathbf{U}_{L_t, \Delta_t}$ and $\mathbf{U}_{L_r, \Delta_r}$ are equal to the M -point and N -point discrete Fourier transform (DFT) matrices, respectively.

The (n, m) entry $g_{n,m}$ of the channel matrix $\mathbf{G}_{\Delta_t, \Delta_r} \in \mathbb{C}^{N \times M}$ in the angular domain is given by

$$g_{n,m} = \sqrt{4L_t L_r} \int a(p) f_{L_r, \Delta_r} \left(\frac{n}{L_r} - \Omega_r(p) \right) \cdot f_{L_t, \Delta_t}^* \left(\frac{m}{L_t} - \Omega_t(p) \right) dp. \quad (24)$$

The prefactor $1/(4\Delta_t \Delta_r) \geq 1$ in the normalized SNR $\tilde{\gamma}$ represents the power gain obtained by spacing antenna elements densely. Since we have ignored a power loss due to mutual coupling, we will exclude the influence of the power gain $1/(4\Delta_t \Delta_r)$ in comparing the critically spaced case and the densely spaced case. In other words, the normalized SNR $\tilde{\gamma}$ will be fixed in comparisons between the two cases. This implies that the actual SNRs γ are different from each other in the two cases.

We consider the large-system limit, in which N , M , L_t , and L_r tend to infinity with the ratios $\Delta_t = L_t/M$, $\Delta_r = L_r/N$, and $\alpha = L_t/L_r$ kept constant. Throughout this paper, we postulate the following for the deterministic channel instance $\mathcal{C} = \{a(\cdot), \Omega_r(\cdot), \Omega_t(\cdot)\}$.

Assumption 1: Let $\mathcal{D}_l = [-(1 - s_l/l), 1 - s_l/l]$, in which $\{s_l \in \mathbb{N}\}$ denotes a slowly diverging sequence of positive integers that satisfies $\lim_{l \rightarrow \infty} s_l = \infty$ and $\lim_{l \rightarrow \infty} s_l/l = 0$. Postulate a class \mathfrak{C} of channel instances (a set of \mathcal{C}) satisfying that $\Omega_r(\cdot) \in \mathcal{D}_{L_r}$ and $\Omega_t(\cdot) \in \mathcal{D}_{L_t}$ hold, and that the total power $\int |a(p)|^2 dp$ of attenuation and the maximum singular value of $\min\{2L_t, 2L_r\}^{-1/2} \mathbf{G}_{\Delta_t, \Delta_r}$ are uniformly bounded in the large-system limit for all channel instances $\mathcal{C} \in \mathfrak{C}$.

The angular domain is restricted in order to use Lemmas 2 and 3. If the channel instances are sampled from proper statistical models, $\Omega_r(\cdot)$ and $\Omega_t(\cdot)$ should be almost surely included into the restricted intervals in the large-system limit.

The bounded maximum singular value implies that we can enjoy no *noiseless* eigen channels in the large-system limit for finite SNR. Thus, this assumption should be satisfied for practical MIMO channels.

IV. MAIN RESULTS

A. Constrained Capacity

Let \mathbf{Q}_M denote an $M \times M$ covariance matrix satisfying the power constraint $\text{Tr}(\mathbf{Q}_M) \leq 1$. We consider precoded Gaussian signaling $\mathbf{x} \sim \mathcal{CN}(\mathbf{0}, \mathbf{Q}_M)$. It is well known that the precoding scheme achieves the constrained capacity of the MIMO channel (20) with the channel matrix \mathbf{H} given by (22),

$$C_{\text{opt}}(\mathbf{Q}_M; \tilde{\gamma}, \mathbf{G}_{\Delta_t, \Delta_r}) = \log \det \left(\mathbf{I} + \tilde{\gamma} \mathbf{G}_{\Delta_t, \Delta_r} \mathbf{U}_{L_t, \Delta_t}^H \mathbf{Q}_M \mathbf{U}_{L_t, \Delta_t} \mathbf{G}_{\Delta_t, \Delta_r}^H \right). \quad (25)$$

The channel capacity is equal to the maximum of (25) over all possible covariance matrices \mathbf{Q}_M satisfying the average power constraint $\text{Tr}(\mathbf{Q}_M) \leq 1$.

The degree of spatial freedom is at most $2\min\{L_t, L_r\}$ [14]. When $\Delta_t, \Delta_r < 1/2$, most of the power of the channel gains $g_{n,m}$ should be concentrated on $n \in \mathcal{N} = [0 : L_r] \cup [N - L_r : N]$ and $m \in \mathcal{M} = [0 : L_t] \cup [M - L_t : M]$ in the large-system limit. For the critically spaced case $\Delta_t = \Delta_r = 1/2$, all channel gains should have significant power.

In order to present a precise statement for this intuition, we consider the densely spaced case $\Delta_t, \Delta_r < 1/2$, and define the $N \times M$ matrix $\tilde{\mathbf{G}}_{\Delta_t, \Delta_r}$ as

$$(\tilde{\mathbf{G}}_{\Delta_t, \Delta_r})_{n,m} = \begin{cases} g_{n,m} & \text{for } (n, m) \in \mathcal{N} \times \mathcal{M}, \\ 0 & \text{otherwise.} \end{cases} \quad (26)$$

Furthermore, it is convenient to define the covariance matrix $\Sigma_M = \mathbf{U}_{L_t, \Delta_t}^H \mathbf{Q}_M \mathbf{U}_{L_t, \Delta_t}$.

Assumption 2: Postulate the set \mathfrak{S}_M of $M \times M$ covariance matrices Σ_M in which the power constraint $\text{Tr}(\Sigma_M) \leq 1$ is satisfied, and in which the maximum eigenvalue of $2L_t \Sigma_M$ is uniformly bounded in the large-system limit.

Theorem 1: Suppose that the precoding matrix is given by $\mathbf{Q}_M = \mathbf{U}_{L_t, \Delta_t} \Sigma_M \mathbf{U}_{L_t, \Delta_t}^H$. Fix SNR $\tilde{\gamma} > 0$, receive antenna separation $\Delta_r \in (0, 1/2]$, and load $\alpha = L_t/L_r > 0$. Under Assumptions 1 and 2, the following limit

$$\frac{|C_{\text{opt}}(\mathbf{Q}_M; \tilde{\gamma}, \mathbf{G}_{\Delta_t, \Delta_r}) - C_{\text{opt}}(\mathbf{Q}_M; \tilde{\gamma}, \tilde{\mathbf{G}}_{\Delta_t, \Delta_r})|}{2\min\{L_t, L_r\}} \rightarrow 0 \quad (27)$$

holds uniformly for all transmit antenna separations $\Delta_t \in (0, 1/2]$, covariance matrices $\Sigma_M \in \mathfrak{S}_M$, and all channel instances $\mathcal{C} \in \mathfrak{C}$ in the large-system limit.

Proof: See Appendix C. ■

Theorem 1 implies that most of the power of the channel gains is concentrated on the antenna indices $\mathcal{N} \times \mathcal{M}$ in terms of the constrained capacity normalized by the degree of spatial freedom. Thus, it is sufficient to consider power allocation over $m \in \mathcal{M}$, as long as the large-system limit is taken. In other words, we consider the $M \times M$ covariance matrix $\mathbf{E}_{L_t+1, M-(2L_t+1)}(\Sigma_{2L_t+1})$ defined via (1) for $\Sigma_{2L_t+1} \in \mathfrak{S}_{2L_t+1}$.

We next make a comparison between the critically spaced case and the densely spaced case with identical L_t and L_r , so that the numbers of antennas are different from each other for the two cases.

Theorem 2: Let $\mathbf{Q}_{2L_t} = \mathbf{U}_{L_t, 1/2} \mathbf{\Sigma}_{2L_t} \mathbf{U}_{L_t, 1/2}^H$ and $\mathbf{Q}_M = \mathbf{U}_{L_t, \Delta_t} \mathbf{E}_{L_t+1, M-(2L_t+1)} (\mathbf{\Sigma}_{2L_t+1}) \mathbf{U}_{L_t, \Delta_t}^H$, defined via (1). Assume that

$$\text{Tr} \left\{ (\mathbf{E}_{L_t, 1} (\mathbf{\Sigma}_{2L_t}) - \mathbf{\Sigma}_{2L_t+1})^2 \right\} \rightarrow 0 \quad (28)$$

as $L_t \rightarrow \infty$. Under Assumptions 1 and 2, for fixed SNR $\tilde{\gamma} > 0$ and load $\alpha > 0$ the following limit

$$\frac{|C_{\text{opt}}(\mathbf{Q}_{2L_t}; \tilde{\gamma}, \mathbf{G}_{1/2, 1/2}) - C_{\text{opt}}(\mathbf{Q}_M; \tilde{\gamma}, \tilde{\mathbf{G}}_{\Delta_t, \Delta_r})|}{2 \min\{L_t, L_r\}} \rightarrow 0 \quad (29)$$

holds uniformly for all antenna separations $\Delta_t, \Delta_r \leq 1/2$, covariance matrices $\mathbf{\Sigma}_{2L_t} \in \mathfrak{S}_{2L_t}$, $\mathbf{\Sigma}_{2L_t+1} \in \mathfrak{S}_{2L_t+1}$, and all channel instances $\mathcal{C} \in \mathfrak{C}$ in the large-system limit.

Proof: See Appendix D. ■

From Theorems 1 and 2, we conclude that the normalized capacity for the critically spaced case is equal to that for the densely spaced case in the large-system limit. In other words, there are no points in using uniform linear antenna arrays with densely spaced antenna elements for all SNRs, as long as optimal Gaussian signaling is assumed. To the best of author's knowledge, Theorems 1 and 2 are the first theoretical results for finite SNRs, although the optimality of the critically spaced case was proved in the high SNR limit [15].

B. Non-Gaussian Signaling

We have so far shown the asymptotic optimality of the critically spaced case for Gaussian signaling. The purpose of this section is to investigate what occurs for suboptimal non-Gaussian signaling, such as QPSK.

Let $\mathbf{b} = (b_0, \dots, b_{M-1})^T$ denote the M -dimensional data symbol vector that has independent QPSK elements $\{b_m\}$ with unit power. For a square root $\mathbf{Q}_M^{1/2}$ of an $M \times M$ covariance matrix $\mathbf{Q}_M \in \mathfrak{S}_M$, the transmitted vector $\mathbf{x} = \mathbf{Q}_M^{1/2} \mathbf{b}$ is generated as the product of the precoding matrix $\mathbf{Q}_M^{1/2}$ and the symbol vector \mathbf{b} . Substituting $\mathbf{x} = \mathbf{Q}_M^{1/2} \mathbf{b}$ and (22) into (20) yields

$$\mathbf{y} = \sqrt{\tilde{\gamma}} \mathbf{A} \mathbf{b} + \mathbf{w}, \quad (30)$$

where the effective channel matrix $\mathbf{A} \in \mathbb{C}^{N \times M}$ is given by

$$\mathbf{A} = \mathbf{U}_{L_r, \Delta_r} \mathbf{G}_{\Delta_t, \Delta_r} \mathbf{U}_{L_t, \Delta_t}^H \mathbf{Q}_M^{1/2}. \quad (31)$$

Then, the achievable rate $C(\mathbf{Q}_M; \tilde{\gamma}, \mathbf{G}_{\Delta_t, \Delta_r})$ of the precoded QPSK scheme is defined as

$$C(\mathbf{Q}_M; \tilde{\gamma}, \mathbf{G}_{\Delta_t, \Delta_r}) = I(\mathbf{b}; \mathbf{y}), \quad (32)$$

where the deterministic channel matrix \mathbf{A} is fixed.

The main result for QPSK is that the achievable rate (32) normalized by the degree of spatial freedom converges to the normalized constrained capacity in the dense limit $\Delta_t \rightarrow 0$ after taking the large-system limit.

Assumption 3: Postulate the set $\tilde{\mathfrak{S}}_M$ of $M \times M$ covariance matrices \mathbf{Q}_M , in which $\tilde{\mathfrak{S}}_M$ is a subset of \mathfrak{S}_M defined in Assumption 2, and in which there exists some constant $A > 0$ such that $\|(2L_t)^{1/2} \mathbf{Q}_M^{1/2} \mathbf{e}_{M,m}\|^2 \leq A \Delta_t$ holds for all $m \in [0 : M)$.

Theorem 3: Under Assumptions 1 and 3, for fixed load $\alpha > 0$ the following limit

$$\frac{|C_{\text{opt}}(\mathbf{Q}_M; \tilde{\gamma}, \mathbf{G}_{\Delta_t, \Delta_r}) - C(\mathbf{Q}_M; \tilde{\gamma}, \mathbf{G}_{\Delta_t, \Delta_r})|}{2 \min\{L_t, L_r\}} \rightarrow 0 \quad (33)$$

holds uniformly for all receive antenna separations $\Delta_r \in (0, 1/2]$, covariance matrices $\mathbf{Q}_M \in \tilde{\mathfrak{S}}_M$, and all channel instances $\mathcal{C} \in \mathfrak{C}$ in the dense limit $\Delta_t \rightarrow 0$ after taking the large-system limit.

Proof: See Section V. ■

Assumption 3 is a sufficient condition under which the linear minimum mean-square error (LMMSE) receiver with successive interference cancellation (SIC) operates in the low signal-to-interference-plus-noise ratio (SINR) regime for all stages. We shall present two examples of the precoding matrix \mathbf{Q}_M satisfying Assumption 3.

Proposition 1: The identity precoding matrix $\mathbf{Q}_M^{1/2} = M^{-1/2} \mathbf{I}_M$ satisfies Assumption 3.

Proof: It is straightforward to confirm $\mathbf{Q}_M \in \mathfrak{S}_M$ and $\|(2L_t)^{1/2} \mathbf{U}_{L_t, \Delta_t}^H \mathbf{Q}_M^{1/2} \mathbf{e}_{M,m}\|^2 = 2\Delta_t$ for all m . ■

Proposition 1 is important for the case in which the transmitter has no ability to perform precoding. This situation may be realistic when low-quality amplifiers are used for all transmit antennas. Theorems 1–3 imply that, when the true SNR $\gamma = 4\Delta_t \Delta_r \tilde{\gamma}$ is considered, equal-power QPSK for $\Delta_r = 1/2$ achieves the normalized constrained capacity for the critically spaced case,

$$\begin{aligned} & (2 \min\{L_t, L_r\})^{-1} C_{\text{opt}}(M^{-1} \mathbf{I}_M; \gamma / (2\Delta_t), \mathbf{G}_{\Delta_t, 1/2}) \\ &= \frac{1}{2 \min\{L_t, L_r\}} \log \det \left(\mathbf{I} + \frac{\gamma}{2L_t} \mathbf{G}_{\Delta_t, 1/2} \mathbf{G}_{\Delta_t, 1/2}^H \right) \\ &= \frac{1}{2 \min\{L_t, L_r\}} C_{\text{opt}}((2L_t)^{-1} \mathbf{I}_{2L_t}; \gamma, \mathbf{G}_{1/2, 1/2}) \end{aligned} \quad (34)$$

in the dense limit $\Delta_t \rightarrow 0$ after taking the large-system limit.

Proposition 2: For any $\Sigma_{2L_t+1} \in \mathfrak{S}_{2L_t+1}$, the precoding matrix $\mathbf{Q}_M^{1/2} = \mathbf{U}_{L_t, \Delta_t} \mathbf{E}_{L_t+1, M-(2L_t+1)} (\Sigma_{2L_t+1}^{1/2}) \mathbf{U}_{L_t, \Delta_t}^H$ satisfies Assumption 3.

Proof: The condition $\Sigma_{2L_t+1} \in \mathfrak{S}_{2L_t+1}$ implies $\mathbf{Q}_M \in \mathfrak{S}_M$. We shall prove $\mathbf{Q}_M \in \tilde{\mathfrak{S}}_M$. Let $\tilde{\mathbf{U}}_{L_t, \Delta_t}$ denote the $M \times (2L_t+1)$ matrix obtained by eliminating the columns $m \notin \mathcal{M}$ from $\mathbf{U}_{L_t, \Delta_t}$. From the definition of $\mathbf{E}_{L_t+1, M-(2L_t+1)} (\Sigma_{2L_t+1}^{1/2})$,

$$\begin{aligned} & \left\| (2L_t)^{1/2} \mathbf{Q}_M^{1/2} \mathbf{e}_{M,m} \right\|^2 \\ &= 2L_t (\tilde{\mathbf{U}}_{L_t, \Delta_t}^H \mathbf{e}_{M,m})^H \Sigma_{2L_t+1} \tilde{\mathbf{U}}_{L_t, \Delta_t}^H \mathbf{e}_{M,m} \\ &\leq \lambda_{\max} \|\tilde{\mathbf{U}}_{L_t, \Delta_t}^H \mathbf{e}_{M,m}\|^2, \end{aligned} \quad (35)$$

where the maximum eigenvalue $\lambda_{\max} > 0$ of $2L_t \Sigma_{2L_t+1}$ is uniformly bounded from Assumption 2. Since the m th column of $\mathbf{U}_{L_t, \Delta_t}$ is equal to $\mathbf{s}_{L_t, \Delta_t}(m/L_t)$ given by (3), we have

$$\|\tilde{\mathbf{U}}_{L_t, \Delta_t}^H \mathbf{e}_{M,m}\|^2 = \frac{\Delta_t}{L_t} \sum_{m' \in \mathcal{M}} |e^{2\pi j m \Delta_t m' / L_t}|^2 \leq 3\Delta_t. \quad (36)$$

Thus, $\mathbf{Q}_M \in \tilde{\mathfrak{S}}_M$ holds. ■

Combining Theorems 1—3, from Proposition 2 we can conclude that the precoded QPSK signaling achieves the normalized channel capacity for the critically spaced case in the dense limit $\Delta_t \rightarrow 0$ after taking the large-system limit, when one excludes the influence of the power gain obtained by spacing antenna elements densely. This result is analogous to the optimality of QPSK FTN signaling as the sampling period tends to zero [18].

C. Discussion

We shall make a comparison between the critically spaced case and the densely spaced case in terms of power consumption in the amplifiers. Since low peak-to-average power ratio (PAPR) results in low power consumption, PAPR is a key factor of the power consumption.

For the critically spaced case, the precoded Gaussian signaling has to be used to achieve the constrained capacity. For the densely spaced case, on the other hand, the precoded QPSK scheme can be used from Theorem 3. For QPSK with no precoding in Proposition 1 the instantaneous power is constant, so that the PAPR is the lowest, while the peak power of Gaussian signaling is unbounded. For the precoded QPSK scheme in Proposition 2, low PAPR is expected from the similarity between the precoding scheme and (localized) subcarrier mapping in single carrier frequency-division multiple access (SC-FDMA) systems [47], [48]. See Section VI for numerical comparisons between the precoded Gaussian and QPSK schemes.

SC-FDMA systems are an alternate low-PAPR scheme of orthogonal frequency-division multiple-access (OFDMA) systems. In SC-FDMA systems, data symbols are converted to the frequency domain with the m -point DFT. After subcarrier mapping, the frequency-domain symbols are re-converted to the time domain via the n -point inverse DFT (IDFT). The precoding scheme in Proposition 2 corresponds to the SC-FDMA scheme with the positions of the DFT and IDFT interchanged, if Σ_{2L_t+1} is diagonal. The only difference is that the sizes of the two transforms are equal to each other in the precoding scheme, while $n > m$ holds in SC-FDMA systems.

V. PROOF OF THEOREM 3

A. Outline

The proof of Theorem 3 consists of two steps. In the former step, a lower bound of the achievable rate (32) is derived on the basis of the LMMSE-SIC [31], [32], [49]. After proving that the SINR in each stage of SIC tends to zero in the dense limit after taking the large-system limit, the interference-plus-noise is replaced with a CSCG random variable by using the fact that, when QPSK is used, the worst-case additive noise in the low SINR regime is Gaussian [50].

In the latter step, we utilize the first-order optimality [51] of each data symbol for the AWGN channel in the low SINR regime to replace the data symbols by optimal Gaussian data symbols. Theorem 3 follows from the optimality of the LMMSE-SIC for Gaussian signaling [52].

B. LMMSE-SIC

We first use the chain rule [19] for the mutual information (32) to obtain

$$I(\mathbf{b}; \mathbf{y}) = \sum_{m=0}^{M-1} I(b_m; \mathbf{y} | b_0, \dots, b_{m-1}), \quad (37)$$

where $I(b_m; \mathbf{y}|b_0, \dots, b_{m-1})$ corresponds to the achievable rate at stage m of SIC based on the optimal minimum mean-square error (MMSE) receiver. Consider the LMMSE estimator \hat{b}_m of b_m based on the known information \mathbf{y} and $\{b_{m'}|m' \in [0 : m]\}$. Since the LMMSE estimator is suboptimal, we have the lower bound

$$I(b_m; \mathbf{y}|b_0, \dots, b_{m-1}) \geq I(b_m; \hat{b}_m|b_0, \dots, b_{m-1}). \quad (38)$$

We shall derive the LMMSE estimator \hat{b}_m . From (30), the output vector at stage m of SIC is given by

$$\mathbf{y} - \sqrt{\tilde{\gamma}} \sum_{m'=0}^{m-1} \mathbf{a}_{m'} b_{m'} = \sqrt{\tilde{\gamma}} \sum_{m'=m}^{M-1} \mathbf{a}_{m'} b_{m'} + \mathbf{w}. \quad (39)$$

In (39), $\mathbf{a}_m \in \mathbb{C}^N$ denotes the m th column vector of the effective channel matrix (31). Thus, the LMMSE estimator \hat{b}_m is given by [42, Eq. (8.66)]

$$\hat{b}_m = \sqrt{\tilde{\gamma}} \mathbf{a}_m^H \mathbf{\Xi}_m \left(\mathbf{y} - \sqrt{\tilde{\gamma}} \sum_{m'=0}^{m-1} \mathbf{a}_{m'} b_{m'} \right), \quad (40)$$

with

$$\mathbf{\Xi}_m = \left(\mathbf{I} + \tilde{\gamma} \sum_{m'=m+1}^{M-1} \mathbf{a}_{m'} \mathbf{a}_{m'}^H \right)^{-1}. \quad (41)$$

We derive the SINR for the LMMSE estimator \hat{b}_m . Define

$$\rho_m = \tilde{\gamma} \mathbf{a}_m^H \mathbf{\Xi}_m \mathbf{a}_m. \quad (42)$$

Substituting (39) into (40) yields

$$\frac{1}{\sqrt{\rho_m}} \hat{b}_m = \sqrt{\rho_m} b_m + v_m, \quad (43)$$

where the interference-plus-noise $v_m \in \mathbb{C}$ is given by

$$\sqrt{\rho_m} v_m = \tilde{\gamma} \sum_{m'=m+1}^{M-1} \mathbf{a}_m^H \mathbf{\Xi}_m \mathbf{a}_{m'} b_{m'} + \sqrt{\tilde{\gamma}} \mathbf{a}_m^H \mathbf{\Xi}_m \mathbf{w}. \quad (44)$$

Since the variance of v_m is equal to 1 from (41), we find that ρ_m is the SINR for the LMMSE estimator \hat{b}_m . Furthermore, (43) implies that the lower bound (38) reduces to

$$I(b_m; \hat{b}_m|b_0, \dots, b_{m-1}) = I(b_m; z_m), \quad (45)$$

with $z_m = \sqrt{\rho_m} b_m + v_m$.

Lemma 4: Fix load $\alpha > 0$. Under Assumptions 1 and 3, there exists some constant $A_\alpha > 0$ such that the multiuser efficiency $\rho_m/(\Delta_t \tilde{\gamma})$ normalized by Δ_t is bounded from above by A_α for all $\Delta_t, \Delta_r \in (0, 1/2]$, covariance matrices $\mathbf{Q}_M \in \tilde{\mathfrak{S}}_M$, channel instances $\mathcal{C} \in \mathfrak{C}$, and all $m \in [0 : M]$.

Proof: Since the maximum eigenvalue of (41) is bounded from above by 1, we have an upper bound for the SINR (42),

$$\frac{\rho_m}{\tilde{\gamma}} = \frac{\mathbf{a}_m^H \mathbf{\Xi}_m \mathbf{a}_m}{\|\mathbf{a}_m\|^2} \|\mathbf{a}_m\|^2 < \left\| \mathbf{G}_{\Delta_t, \Delta_r} \mathbf{U}_{L_t, \Delta_t}^H \mathbf{Q}_M^{1/2} \mathbf{e}_{M,m} \right\|^2, \quad (46)$$

where we have used the fact that \mathbf{a}_m is the m th column of (31). Repeating the same argument yields

$$\frac{\rho_m}{\tilde{\gamma}} < \sigma_{\max}^2 \left\| (2L_t)^{1/2} \mathbf{Q}_M^{1/2} \mathbf{e}_{M,m} \right\|^2, \quad (47)$$

where $\sigma_{\max} > 0$ denotes the maximum singular value of the channel matrix $(2L_t)^{-1/2} \mathbf{G}_{\Delta_t, \Delta_r}$. From Assumptions 1 and 3, we find that Lemma 4 holds. ■

A further lower bound for (45) is derived by using the fact that Gaussian noise is the worst-case additive noise for the real AWGN channel with binary phase-shift keying (BPSK) [50]. Since the real and imaginary parts $\Re[b_m]$ and $\Im[b_m]$ are independent for QPSK, (45) reduces to

$$I(b_m; z_m) = H(\Re[b_m]) + H(\Im[b_m]) - H(b_m|z_m). \quad (48)$$

Using the chain rule for entropy [19] yields

$$\begin{aligned} H(b_m|z_m) &= H(\Re[b_m]|z_m) + H(\Im[b_m]|z_m, \Re[b_m]) \\ &\leq H(\Re[b_m]|\Re[z_m]) + H(\Im[b_m]|\Im[z_m]), \end{aligned} \quad (49)$$

where the inequality follows from the fact that conditioning reduces entropy [19]. Thus, we have

$$I(b_m; z_m) \geq I(\Re[b_m]; \Re[z_m]) + I(\Im[b_m]; \Im[z_m]). \quad (50)$$

Replacing $\Re[v_m]$ and $\Im[v_m]$ in z_m with the worst-case additive noise for the BPSK AWGN channel in the low SNR regime [50], we arrive at the lower bound,

$$I(b_m; z_m) \geq I(b_m; \sqrt{\rho_m} b_m + v_m^G), \quad (51)$$

with $v_m^G \sim \mathcal{CN}(0, 1)$, in the dense limit $\Delta_t \rightarrow 0$ after taking the large-system limit uniformly for all $\Delta_r \in (0, 1/2]$, covariance matrices $\mathbf{Q}_M \in \tilde{\mathfrak{S}}_M$, and all channel instances $\mathcal{C} \in \mathfrak{C}$. Applying (38), (45), and (51) to (37) yields

$$\frac{1}{M} I(\mathbf{b}; \mathbf{y}) \geq \frac{1}{M} \sum_{m=0}^{M-1} I(b_m; \sqrt{\rho_m} b_m + v_m^G) \quad (52)$$

in the dense limit after taking the large-system limit.

Remark 1: One may attempt to use the central limit theorem in order to replace the interference-plus-noise v_m by the CSCG random variable v_m^G in the large-system limit. The asymptotic Gaussianity of the interference was proved in [53] when the channel matrix \mathbf{H} has i.i.d. elements. However, it is not clear whether there exists a pathological channel instance $\mathcal{C} \in \mathfrak{C}$ such that v_m is non-Gaussian even in the large-system limit. If v_m converges to v_m^G , we could postulate quadrature amplitude modulation (QAM) data symbols to obtain Theorem 3, instead of QPSK.

C. Optimality of LMMSE-SIC

We use the first-order optimality of b_m to evaluate the mutual information $I(b_m; \sqrt{\rho_m} b_m + v_m^G)$. Let $b_m^G \sim \mathcal{CN}(0, 1)$ denote a CSCG data symbol with unit power. Since any zero-mean and unit-power signaling is first-order optimal for the AWGN channel [51, Theorem 4], we have

$$\frac{|I(b_m; \sqrt{\rho_m} b_m + v_m^G) - I(b_m^G; \sqrt{\rho_m} b_m^G + v_m^G)|}{\rho_m} \rightarrow 0 \quad (53)$$

uniformly for all $\Delta_r \in (0, 1/2]$, covariance matrices $\mathbf{Q}_M \in \tilde{\mathfrak{S}}_M$, and all channel instances $\mathcal{C} \in \mathfrak{C}$ in the dense limit $\Delta_t \rightarrow 0$ after taking the large-system limit.

Applying (53) to (52), from Lemma 4 we find

$$\frac{1}{M} I(\mathbf{b}; \mathbf{y}) \geq \frac{1}{M} C_{\text{opt}}(\mathbf{Q}_M; \tilde{\gamma}, \mathbf{G}_{\Delta_t, \Delta_r}) + o(\Delta_t), \quad (54)$$

in the dense limit $\Delta_t \rightarrow 0$ after taking the large-system limit. In the derivation of (54), we have used the fact that the LMMSE-SIC for Gaussian signaling achieves the constrained capacity $C_{\text{opt}}(\mathbf{Q}_M; \tilde{\gamma}, \mathbf{G}_{\Delta_t, \Delta_r})$ given by (25) [52]. Dividing both sides by Δ_t , we have

$$\frac{L_t}{2\min\{L_t, L_r\}} \frac{I(\mathbf{b}; \mathbf{y}) - C_{\text{opt}}(\mathbf{Q}_M; \tilde{\gamma}, \mathbf{G}_{\Delta_t, \Delta_r})}{L_t} \geq 0 \quad (55)$$

in the dense limit $\Delta_t \rightarrow 0$ after taking the large-system limit. Since the upper bound $C_{\text{opt}}(\mathbf{Q}_M; \tilde{\gamma}, \mathbf{G}_{\Delta_t, \Delta_r}) - I(\mathbf{b}; \mathbf{y}) \geq 0$ is trivial, we arrive at Theorem 3.

VI. NUMERICAL RESULTS

A. Simulation Conditions

In all numerical results, we assume the P -path Rayleigh fading model as an example of the channel matrix $\mathbf{G}_{\Delta_t, \Delta_r}$ in the angular domain—given by (24),

$$(\mathbf{G}_{\Delta_t, \Delta_r})_{n,m} = \sqrt{4L_t L_r} \sum_{p=0}^{P-1} a_p f_{L_r, \Delta_r} \left(\frac{n}{L_r} - \cos \phi_{r,p} \right) \cdot f_{L_t, \Delta_t}^* \left(\frac{m}{L_t} - \cos \phi_{t,p} \right). \quad (56)$$

In (56), the attenuation $\{a_p | p \in [0 : P]\}$ are independently sampled from the CSCG distribution with variance $1/P$. Furthermore, the angles $\{\phi_{t,p}, \phi_{r,p} | p \in [0 : P]\}$ of departure and incident are independently drawn from the uniform distribution on $[0, 2\pi)$. The condition $P \geq \min\{2L_t, 2L_r\}$ is necessary for achieving the full spatial degrees of freedom.

We focus on single-user massive MIMO downlink, in which the transmitter is a base station with a large antenna array, while the receiver corresponds to a user with a small antenna array. Thus, we consider the case in which the normalized length L_t of the transmit antenna array is larger than the normalized length L_r of the receive antenna array. The transmitter may

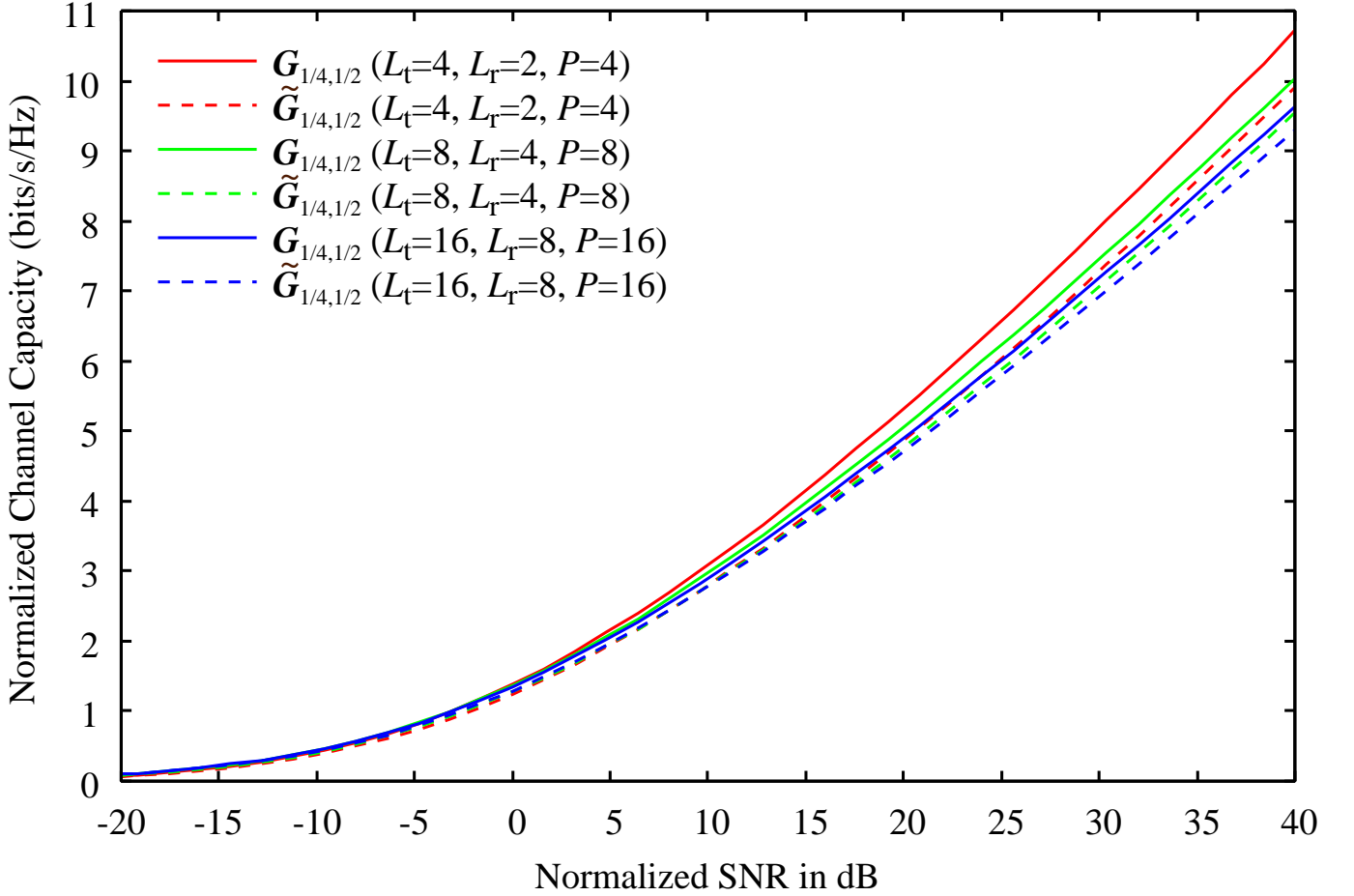


Fig. 2. Normalized channel capacity C_{opt}/N versus normalized SNR $\tilde{\gamma}$ under the full CSI assumption for $\Delta_t = 1/4$ and $\Delta_r = 1/2$.

equip densely spaced antennas ($\Delta_t \leq 1/2$), whereas the receiver has critically spaced antennas ($\Delta_r = 1/2$). Thus, the number $M = L_t/\Delta_t$ of transmit antennas may be much larger than the number $N = L_r/\Delta_r$ of receive antennas.

Consider the precoding scheme $\mathbf{x} = \mathbf{Q}_M^{1/2} \mathbf{b}$, with $\mathbf{Q}_M = \mathbf{U}_{L_t, \Delta_t} \Sigma_M \mathbf{U}_{L_t, \Delta_t}^H$. In estimating the constrained capacity ($\mathbf{b} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$), we consider the ergodic constrained capacity averaged over sufficiently many independent channel instances, called constrained capacity simply. For the case of full channel state information (CSI), the input covariance matrix $\Sigma_M \in \mathfrak{S}_M$ was optimized with the water-filling algorithm [42]. For the case of CSI at the receiver (CSIR), we considered the two diagonal input covariance matrices $\Sigma_{2L_t} = (2L_t)^{-1} \mathbf{I}_{2L_t}$ for the critically spaced case $\Delta_t = 1/2$ and $\Sigma_M = (2L_t + 1)^{-1} \text{diag}\{\mathbf{1}_{L_t+1}, \mathbf{0}, \mathbf{1}_{L_t}\}$ for the densely spaced case $\Delta_t < 1/2$, in which $\mathbf{1}_n$ denotes the n -dimensional vector whose elements are all one. In the densely spaced case, the power is uniformly allocated to the antenna indices in which the transmit beamforming patterns have main lobes.

B. Precoded Gaussian Signaling

Consider the case of Gaussian signaling. We first investigate the influence of channel gains for indices $m \in \mathcal{M}^c = (L_t : M - L_t)$, in which the transmit beamforming patterns have no main lobes, as shown in Fig. 1.

Figure 2 shows the normalized channel capacities for the two channel matrices $\mathbf{G}_{\Delta_t, 1/2}$ and $\tilde{\mathbf{G}}_{\Delta_t, 1/2}$ given by (26) under the assumption of full channel state information (CSI). We find that there are gaps between the two capacities at high SNRs. The gap decreases slowly as L_t and L_r grows at the same rate. This result is consistent² with Theorem 1.

We next make comparisons between the critically spaced case $\Delta_t = 1/2$ and the densely spaced case $\Delta_t < 1/2$. Figure 3 shows the constrained capacities for the two channel matrices $\mathbf{G}_{1/2, 1/2}$ and $\mathbf{G}_{\Delta_t, 1/2}$ under the CSIR assumption. We find that the two capacities are indistinguishable from each other even for small systems. This result implies that the difference (29) in Theorem 2 converges very quickly on average, while the proof of Theorem 2 predicts slow convergence for the worst channel instance.

² Numerical simulations showed that the average of the individual differences for identical channel instances decreased slowly, although no figure is presented.

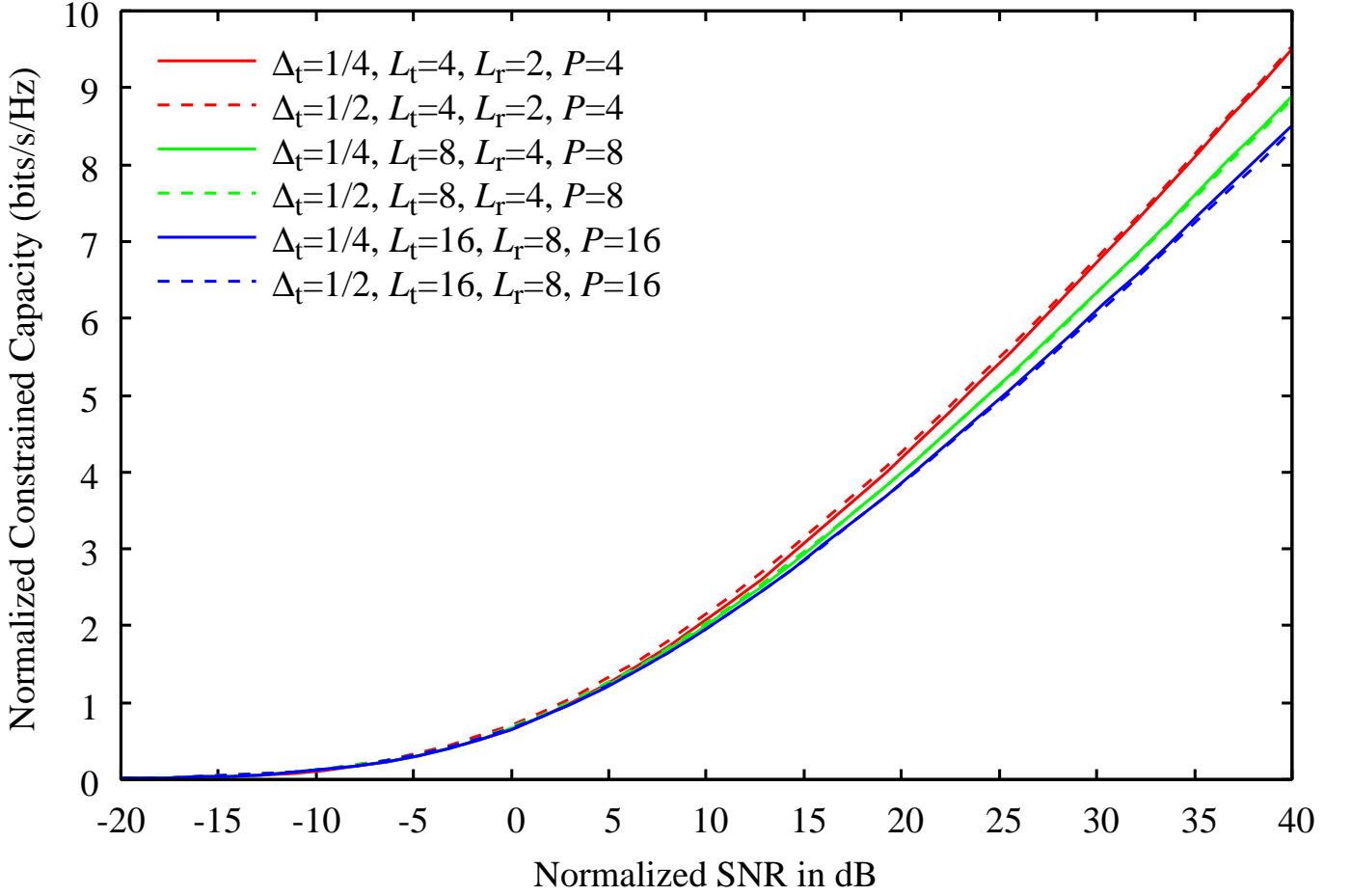


Fig. 3. Normalized constrained capacity C_{opt}/N versus normalized SNR $\tilde{\gamma}$ under the CSIR assumption for $\Delta_r = 1/2$.

C. Precoded Non-Gaussian Signaling

VII. CONCLUSION

APPENDIX A

PROOF OF LEMMA 2

In proving Lemma 2, we use the following lemma.

Lemma 5:

$$\sum_{k=n+1}^{\infty} \frac{1}{k^2} = O(n^{-1}) \quad \text{as } n \rightarrow \infty. \quad (57)$$

Proof: The lemma follows from the bound

$$\frac{\pi^2}{6} \frac{2n(2n-1)}{(2n+1)^2} < \sum_{k=1}^n \frac{1}{k^2} < \frac{\pi^2}{6} \frac{2n(2n+2)}{(2n+1)^2}. \quad (58)$$

We review a classical elementary proof of (58). Start with the identity for $2n+1 \in \mathbb{N}$

$$\frac{e^{j(2n+1)x}}{(\sin x)^{2n+1}} = \left(\frac{1}{\tan x} + j \right)^{2n+1}. \quad (59)$$

Binomial-expanding the right-hand side (RHS) and then comparing the imaginary parts on both sides, we obtain

$$\frac{\sin\{(2n+1)x\}}{(\sin x)^{2n+1}} = \sum_{r=0}^n \binom{2n+1}{2r+1} (-1)^r t^{n-r}, \quad (60)$$

with $t = (\tan x)^{-2}$.

Consider $x_k = k\pi/(2n+1)$ for $k = 1, \dots, n$. It is straightforward to confirm that $\{(\tan x_k)^{-2} | k \in [1 : n]\}$ are the roots of the n th-degree polynomial on the RHS of (60) with respect to t . From Vieta's formulas, we have

$$\sum_{k=1}^n \frac{1}{\tan^2 x_k} = \binom{2n+1}{1}^{-1} \binom{2n+1}{3} = \frac{2n(2n-1)}{6}. \quad (61)$$

Applying the identity $(\sin x_k)^{-2} = 1 + (\tan x_k)^{-2}$ yields

$$\sum_{k=1}^n \frac{1}{\sin^2 x_k} = \frac{2n(2n+2)}{6}. \quad (62)$$

The bound (58) follows from (61), (62), and the bound $\sin^2 x_k < x_k^2 < \tan^2 x_k$ for all $x_k = k\pi/(2n+1)$. \blacksquare

Let us prove Lemma 2. From $\mathcal{K}_2 = \emptyset$ for $\Delta = 1/2$, by definition $I_2(\Omega, \Omega'; L, \Delta) = 0$ holds. Thus, we assume $\Delta < 1/2$. For notational simplicity, we write $f_{L,\Delta}$ as f .

Using the Cauchy-Schwarz inequality, we have an upper-bound for $|I_2(\Omega, \Omega'; L, \Delta)|^2$ given by (14),

$$|I_2(\Omega, \Omega'; L, \Delta)|^2 \leq F(\Omega)F(\Omega'), \quad (63)$$

with

$$F(\Omega) = \sum_{k=L+1}^{N-L-1} \left| f\left(\frac{k}{L} - \Omega\right) \right|^2. \quad (64)$$

Thus, it is sufficient to prove that $s_L F(\Omega)$ is uniformly bounded for all $\Omega \in \mathcal{D}_L$ and $\Delta \in (0, 1/2)$.

Since $|f(\Omega)|$ is an even periodic function with period N/L from Property 1, i.e. $|f(\Omega)| = |f(N/L - \Omega)|$, we use the property $N = \lceil N/2 \rceil + \lfloor N/2 \rfloor$ to represent $F(\Omega)$ as

$$\begin{aligned} F(\Omega) &= \sum_{k=L+1}^{\lceil N/2 \rceil - 1} \left| f\left(\frac{k}{L} - \Omega\right) \right|^2 + \sum_{k=L+1}^{\lfloor N/2 \rfloor} \left| f\left(\frac{k}{L} + \Omega\right) \right|^2 \\ &\leq \sum_{k=L+1}^{\lfloor N/2 \rfloor} \left\{ \left| f\left(\frac{k}{L} - \Omega\right) \right|^2 + \left| f\left(\frac{k}{L} + \Omega\right) \right|^2 \right\}. \end{aligned} \quad (65)$$

Since the upper bound (65) is an even function of Ω , without loss of generality, we assume $\Omega \in [0, 1 - s_L/L]$. In order to use the upper bound (7), we decompose (65) into two terms,

$$F(\Omega) \leq F_1(\Omega) + F_2(\Omega), \quad (66)$$

with

$$F_1 = \sum_{k=L+1}^{\lfloor N/2 \rfloor} \left| f\left(\frac{k}{L} - \Omega\right) \right|^2 + \sum_{k=L+1}^{\lceil N/2 - L\Omega \rceil - 1} \left| f\left(\frac{k}{L} + \Omega\right) \right|^2, \quad (67)$$

$$F_2 = \sum_{k=\lceil N/2 - L\Omega \rceil}^{\lfloor N/2 \rfloor} \left| f\left(\frac{k}{L} + \Omega\right) \right|^2. \quad (68)$$

Note that $k + L\Omega \geq N/2$ holds in $F_2(\Omega)$ when k runs from $\lceil N/2 - L\Omega \rceil$.

We first upper-bound $F_1(\Omega)$. Substituting the expression (5) and using the upper bound (7), we find

$$\begin{aligned} F_1(\Omega) &< \frac{1}{4} \sum_{k=L+1}^{\lfloor N/2 \rfloor} \frac{1}{(k - L\Omega)^2} + \frac{1}{4} \sum_{k=L+1}^{\lceil N/2 - L\Omega \rceil - 1} \frac{1}{(k + L\Omega)^2} \\ &< \frac{1}{4} \sum_{k=s_L+1}^{\infty} \frac{1}{k^2} + \frac{1}{4} \sum_{k=L+1}^{\infty} \frac{1}{k^2}, \end{aligned} \quad (69)$$

for all $\Omega \in [0, 1 - s_L/L]$.

We next evaluate $F_2(\Omega)$. Property 1 implies the symmetry $|f(\Omega)| = |f(\Omega - 1/\Delta)| = |f(1/\Delta - \Omega)|$. From $k + L\Omega \geq N/2$ for $k \geq \lceil N/2 - L\Omega \rceil$, we use the upper bound (7) to obtain

$$\begin{aligned} F_2(\Omega) &= \sum_{k=\lceil N/2-L\Omega \rceil}^{\lfloor N/2 \rfloor} |\text{sinc}_N(N - k - L\Omega)|^2 \\ &< \frac{1}{4} \sum_{k=\lceil N/2-L\Omega \rceil}^{\lfloor N/2 \rfloor-1} \frac{1}{(N - k - L\Omega)^2} + |\text{sinc}_N(\lceil N/2 \rceil - L\Omega)| \\ &< \frac{1}{4} \sum_{k=\lceil N/2 \rceil-L+s_L}^{\infty} \frac{1}{k^2} + \frac{1}{4\lfloor N/2 \rfloor^2}, \end{aligned} \quad (70)$$

for all $\Omega \in [0, 1 - s_L/L]$. In the derivation of the last inequality, we have used the following upper bounds:

$$\begin{aligned} |\text{sinc}_N(\lceil N/2 \rceil - L\Omega)| &\leq \frac{1}{2\lceil N/2 \rceil - L\Omega} \\ &\leq \frac{1}{2\lceil N/2 \rceil - L + s_L}, \end{aligned} \quad (71)$$

for all $\lceil N/2 \rceil - L\Omega \leq N/2$. Otherwise,

$$|\text{sinc}_N(\lceil N/2 \rceil - L\Omega)| = |\text{sinc}_N(\lfloor N/2 \rfloor + L\Omega)| \leq \frac{1}{2\lfloor N/2 \rfloor}. \quad (72)$$

From Lemma 5, (66), (69), and (70), we find that $s_L F(\Omega)$ is uniformly bounded for all $\Omega \in \mathcal{D}_L$ and $\Delta \in (0, 1/2)$ in the limit $L, N \rightarrow \infty$ with $\Delta = L/N$ fixed.

APPENDIX B PROOF OF LEMMA 3

For notational convenience, $D_{k,k}(\Omega)$ is simply written as $D_k(\Omega)$. We first prove the following lemma.

Lemma 6: There exists some constant $A > 0$ such that

$$|D_k(\Omega)|^2 < \frac{1}{4L^2} + \frac{A}{(2L - |k - L\Omega|)^2}, \quad (73)$$

for all $k \in [0 : L]$, $\Omega \in \mathcal{D}_L$, and all $\Delta \in (0, 1/2]$.

Proof: Let $x = k - L\Omega \in [-L + s_L, 2L - s_L]$. From (5), (6), and (18), we have

$$|D_k(\Omega)|^2 = \sin^2(\pi x) \left\{ |d_\Delta(x)|^2 + \frac{1}{L^2} \left(\frac{1}{2} - \Delta \right)^2 \right\}, \quad (74)$$

with

$$d_\Delta(x) = \frac{1}{L} \left\{ \frac{\Delta}{\tan(\pi \Delta x / L)} - \frac{1}{2 \tan(\pi x / (2L))} \right\}, \quad (75)$$

where we define $d_\Delta(0) = \lim_{x \rightarrow 0} d_\Delta(x) = 0$.

The upper bound (73) follows from (74) and the following bound:

$$|d_\Delta(x)| < \bar{d}(x) = \frac{\sqrt{A}}{2L - |x|}, \quad \text{for } x \in (-2L, 2L). \quad (76)$$

for some constant $A > 0$.

Let us prove the upper bound (76). Since $|d_\Delta(x)|$ is an even function, we only consider the interval $[0, 2L)$. It is straightforward to confirm $d_\Delta(x) \geq 0$ for $x \in [0, 2L)$ and $\Delta \in (0, 1/2]$, because $u/\tan u$ is monotonically decreasing for $u \in [0, \pi)$. Furthermore, $d_\Delta(x) \leq d_{\Delta'}(x)$ for $\Delta' \leq \Delta$ holds. Taking the limit $\Delta \rightarrow 0$, we have the upper bound $d_\Delta(x) \leq d_0(x)$, with

$$d_0(x) = \frac{1}{\pi x} - \frac{1}{2L \tan(\pi x / (2L))}. \quad (77)$$

Let $y = \pi x / (2L) \in [0, \pi)$. Evaluating the product $\pi(2L - x)d_\Delta(x)$ yields

$$\pi(2L - x)d_\Delta(x) \leq \frac{\pi - y}{y} - \frac{\pi - y}{\tan y} \equiv \tilde{d}_0(y). \quad (78)$$

The boundedness of $\tilde{d}_0(y)$ follows from $\lim_{y \rightarrow 0} \tilde{d}_0(y) = 0$, $\lim_{y \rightarrow \pi} \tilde{d}_0(y) = 1$, and the continuity of $\tilde{d}_0(y)$ on $(0, \pi)$. Thus, the upper bound (76) holds. \blacksquare

Lemma 3 can be proved in the same manner as in the proof of Lemma 2, by using Lemma 6 instead of (7). Using the Cauchy-Schwarz inequality, we upper-bound the quantity $|I_3(\Omega, \Omega'; L, \Delta)|^2$ given by (17) as

$$|I_3(\Omega, \Omega'; L, \Delta)|^2 \leq G(\Omega)G(\Omega'), \quad (79)$$

with

$$G(\Omega) = \sum_{k=0}^{2L-1} |D_{k, \kappa_{L, \Delta}(k)}(\Omega)|^2 + |f_{L, \Delta}(1 - \Omega)|^2. \quad (80)$$

Thus, it is sufficient to prove that $s_L G(\Omega)$ is uniformly bounded for all $\Omega \in \mathcal{D}_L$ and $\Delta \in (0, 1/2]$.

From (5), (16), (18), and the first two properties in Property 1, we have

$$G(\Omega) = \sum_{k=0}^{L-1} |D_k(\Omega)|^2 + \sum_{k=1}^L |D_k(-\Omega)|^2 + |\text{sinc}_{L/\Delta}(L(1 - \Omega))|^2. \quad (81)$$

Applying the upper bound (73) to (81) yields

$$G(\Omega) < \frac{1}{2L} + AS(\Omega) + |\text{sinc}_{L/\Delta}(L(1 - \Omega))|^2, \quad (82)$$

with

$$S(\Omega) = \sum_{k=0}^L \left\{ \frac{1}{(2L - |k - L\Omega|)^2} + \frac{1}{(2L - |k + L\Omega|)^2} \right\}. \quad (83)$$

We first evaluate $|\text{sinc}_{L/\Delta}(L(1 - \Omega))|$. When $L(1 - \Omega) \leq N/2$ holds, from the upper bound (7) we have

$$|\text{sinc}_{L/\Delta}(L(1 - \Omega))| \leq \frac{1}{2L(1 - \Omega)} \leq \frac{1}{2s_L}, \quad (84)$$

for all $\Omega \in \mathcal{D}_L$. Otherwise,

$$\begin{aligned} |\text{sinc}_{L/\Delta}(L(1 - \Omega))| &= |\text{sinc}_{L/\Delta}(N - L(1 - \Omega))| \\ &\leq \frac{1}{2(N - 2L + s_L)}. \end{aligned} \quad (85)$$

Combining the two upper bounds yields

$$|\text{sinc}_{L/\Delta}(L(1 - \Omega))| \leq \frac{1}{2s_L}, \quad (86)$$

for all $\Omega \in \mathcal{D}_L$ and $\Delta \in (0, 1/2]$.

We next evaluate (83). Since $S(\Omega)$ is an even function of $\Omega \in \mathcal{D}_L$, without loss of generality, $\Omega \in [0, 1 - s_L/L]$ is assumed. We decompose $S(\Omega)$ into the sum $S_1(\Omega) + S_2(\Omega)$, with

$$S_1(\Omega) = \sum_{k=0}^{\lfloor L\Omega \rfloor} \frac{1}{(2L + k - L\Omega)^2} + \sum_{k=0}^L \frac{1}{\{2L - (k + L\Omega)\}^2}, \quad (87)$$

$$S_2(\Omega) = \sum_{k=\lfloor L\Omega \rfloor + 1}^L \frac{1}{\{2L - (k - L\Omega)\}^2}. \quad (88)$$

For $S_1(\Omega)$, we have

$$\begin{aligned} S_1(\Omega) &\leq \sum_{k=0}^{\lfloor L\Omega \rfloor} \frac{1}{(L + s_L + k)^2} + \sum_{k=0}^L \frac{1}{(L + s_L - k)^2} \\ &< \sum_{k=L+s_L}^{\infty} \frac{1}{k^2} + \sum_{k=s_L}^{\infty} \frac{1}{k^2}. \end{aligned} \quad (89)$$

On the other hand, for $S_2(\Omega)$

$$S_2(\Omega) \leq \sum_{k=\lfloor L\Omega \rfloor + 1}^L \frac{1}{(2L - k)^2} < \sum_{k=L}^{\infty} \frac{1}{k^2}. \quad (90)$$

Combining (82), (86), (89), and (90), from Lemma 5 we find that $s_L G(\Omega)$ is uniformly bounded for all $\Omega \in \mathcal{D}_L$ and $\Delta \in (0, 1/2]$ in the limit $L, N \rightarrow \infty$ with $\Delta = L/N$ fixed.

APPENDIX C PROOF OF THEOREM 1

A. Asymptotically Equivalent Matrices

In order to prove Theorem 1, we first introduce the notion of asymptotically equivalent matrices [43, Chapter 2]. After defining two norms on the space of $N \times M$ complex matrices, we present the definition of asymptotically equivalent matrices.

Definition 1: The operator norm of $\mathbf{A} \in \mathbb{C}^{N \times M}$ is defined as

$$\|\mathbf{A}\| = \sup_{\mathbf{x} \in \mathbb{C}^M: \mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|}. \quad (91)$$

Definition 2: The normalized Frobenius norm of $\mathbf{A} \in \mathbb{C}^{N \times M}$ is defined as

$$\|\mathbf{A}\|_F = \sqrt{N^{-1} \text{Tr}(\mathbf{A}\mathbf{A}^H)}. \quad (92)$$

Definition 3: Two matrices $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{N \times M}$ are called asymptotically equivalent matrices if the operator norms $\|\mathbf{A}\|$ and $\|\mathbf{B}\|$ are uniformly bounded for all N and M , and if the normalized Frobenius norm $\|\mathbf{A} - \mathbf{B}\|_F$ tends to zero in the large matrix limit $N, M \rightarrow \infty$ with $\beta = M/N$ kept constant.

The purpose of this section is to prove the following theorem:

Theorem 4 ([43]): For two positive semi-definite Hermitian matrices $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{N \times N}$, let $\{\lambda_n \geq 0\}$ and $\{\tilde{\lambda}_n \geq 0\}$ denote the eigenvalues of \mathbf{A} and \mathbf{B} , respectively. If \mathbf{A} and \mathbf{B} are asymptotically equivalent, then

$$J = \frac{1}{N} \sum_{n=0}^{N-1} \left\{ \psi(\lambda_n) - \psi(\tilde{\lambda}_n) \right\} \rightarrow 0, \quad (93)$$

in the large matrix limit, for any continuous function ψ .

In order to prove Theorem 4, we first present several properties of the two norms.

Property 2:

- The operator norm $\|\mathbf{A}\|$ is equal to the maximum singular value of \mathbf{A} . Furthermore, $\|\mathbf{A}_1\| \leq \|\mathbf{A}\|$ for any submatrix \mathbf{A}_1 of \mathbf{A} .
- The operator norm $\|\cdot\|$ is submultiplicative,

$$\|\mathbf{A}\mathbf{B}\| \leq \|\mathbf{A}\| \|\mathbf{B}\|. \quad (94)$$

- For $\mathbf{A} \in \mathbb{C}^{N \times L}$ and $\mathbf{B} \in \mathbb{C}^{L \times M}$,

$$\|\mathbf{A}\mathbf{B}\|_F \leq \sqrt{\frac{L}{N}} \|\mathbf{A}\| \|\mathbf{B}\|_F. \quad (95)$$

- For the square case $N = M$,

$$\frac{1}{N} |\text{Tr}(\mathbf{A})| \leq \|\mathbf{A}\|_F. \quad (96)$$

Proof: The former statement in the first property is well known. In proving the latter statement, without loss of generality, we assume

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{pmatrix}. \quad (97)$$

Let M_1 denote the number of columns of the submatrix \mathbf{A}_1 . By restricting the m th elements of \mathbf{x} in (91) to zero for $m \geq M_1$, we obtain the lower bound

$$\begin{aligned} \|\mathbf{A}\| &\geq \sup_{\tilde{\mathbf{x}} \in \mathbb{C}^{M_1}: \tilde{\mathbf{x}} \neq \mathbf{0}} \frac{1}{\|\tilde{\mathbf{x}}\|} \left\| \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_3 \end{pmatrix} \tilde{\mathbf{x}} \right\| \\ &= \sup_{\mathbf{y} \in \mathbb{C}^N: \mathbf{y} \neq \mathbf{0}} \frac{\|(\mathbf{A}_1^T, \mathbf{A}_3^T)\mathbf{y}\|}{\|\mathbf{y}\|}. \end{aligned} \quad (98)$$

Repeating the same argument yields $\|\mathbf{A}\| \geq \|\mathbf{A}_1\|$.

The second property is trivial for $\mathbf{B} = \mathbf{O}$. For non-zero matrices \mathbf{B} , it follows from

$$\|\mathbf{A}\mathbf{B}\| = \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{A}(\mathbf{B}\mathbf{x})\|}{\|\mathbf{B}\mathbf{x}\|} \frac{\|\mathbf{B}\mathbf{x}\|}{\|\mathbf{x}\|} \leq \sup_{\mathbf{y} \neq \mathbf{0}} \frac{\|\mathbf{A}\mathbf{y}\|}{\|\mathbf{y}\|} \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{B}\mathbf{x}\|}{\|\mathbf{x}\|}. \quad (99)$$

We shall show the third property. Let \mathbf{b}_m denote the m th non-zero column of \mathbf{B} . From the definition of the normalized Frobenius norm,

$$\begin{aligned}\|\mathbf{AB}\|_{\text{F}}^2 &= \frac{1}{N} \sum_m \frac{\|\mathbf{Ab}_m\|^2}{\|\mathbf{b}_m\|^2} \|\mathbf{b}_m\|^2 \\ &\leq \frac{1}{N} \|\mathbf{A}\|^2 \sum_m \|\mathbf{b}_m\|^2 \\ &= \frac{L}{N} \|\mathbf{A}\|^2 \|\mathbf{B}\|_{\text{F}}^2,\end{aligned}\tag{100}$$

where the inequality is due to the definition of the operator norm.

Finally, we prove the last property for $N = M$. The trace $\text{Tr}(\mathbf{A})$ can be regarded as the inner product of \mathbf{A} and \mathbf{I}_N . We use the Cauchy-Schwarz inequality to obtain

$$\frac{1}{N} |\text{Tr}(\mathbf{A})| \leq \frac{1}{N} \sqrt{\text{Tr}(\mathbf{AA}^{\text{H}})} \sqrt{\text{Tr}(\mathbf{I}_N^2)} = \|\mathbf{A}\|_{\text{F}}.\tag{101}$$

■

In proving Theorem 4, we use the following lemma for asymptotically equivalent matrices.

Lemma 7: If $\mathbf{A} \in \mathbb{C}^{N \times L}$ and $\mathbf{B} \in \mathbb{C}^{N \times L}$ are asymptotically equivalent and if $\mathbf{C} \in \mathbb{C}^{L \times M}$ and $\mathbf{D} \in \mathbb{C}^{L \times M}$ are asymptotically equivalent, \mathbf{AC} and \mathbf{BD} are also asymptotically equivalent.

Proof: The boundedness of the operator norms $\|\mathbf{AC}\|$ and $\|\mathbf{BD}\|$ follows from the second property in Property 2.

For the normalized Frobenius norm, we use the triangle inequality to obtain

$$\begin{aligned}\|\mathbf{AC} - \mathbf{BD}\|_{\text{F}} &\leq \|\mathbf{A}(\mathbf{C} - \mathbf{D})\|_{\text{F}} + \|(\mathbf{A} - \mathbf{B})\mathbf{D}\|_{\text{F}} \\ &\leq \sqrt{\frac{L}{N}} \|\mathbf{A}\| \|\mathbf{C} - \mathbf{D}\|_{\text{F}} + \sqrt{\frac{L}{M}} \|\mathbf{D}\| \|(\mathbf{A} - \mathbf{B})\|_{\text{F}},\end{aligned}\tag{102}$$

where the last inequality is due to the third property in Property 2. The bound (102) implies $\|\mathbf{AC} - \mathbf{BD}\|_{\text{F}} \rightarrow 0$.

■

We are ready to prove Theorem 4.

Proof of Theorem 4: We first show Theorem 4 for the case $\psi(x) = x^k$ with non-negative integers k . Since $\{\lambda_n^k\}$ and $\{\tilde{\lambda}_n^k\}$ are the eigenvalues of \mathbf{A}^k and \mathbf{B}^k , (93) reduces to

$$|J| = \frac{1}{N} \left| \text{Tr}(\mathbf{A}^k - \mathbf{B}^k) \right| \leq \left\| \mathbf{A}^k - \mathbf{B}^k \right\|_{\text{F}},\tag{103}$$

where we have used the last property in Property 2. Using Lemma 7 repeatedly, we find that \mathbf{A}^k and \mathbf{B}^k are asymptotically equivalent. Thus, (103) tends to zero.

It is straightforward to confirm that Theorem 4 is correct for any polynomial $\psi(x)$. For the general case, we use the Weierstrass approximation theorem [43]. Since $\{\lambda_n\}$ and $\{\tilde{\lambda}_n\}$ are bounded, the domain of the continuous function ψ can be restricted to an interval $[0, \lambda_{\max}]$. Thus, for any $\epsilon > 0$ there exists some polynomial $p(x)$ such that

$$\sup_{x \in [0, \lambda_{\max}]} |\psi(x) - p(x)| < \frac{\epsilon}{3}.\tag{104}$$

For these ϵ and $p(x)$, we have proved that there are some $N_0 \in \mathbb{N}$ such that

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} \left\{ p(\lambda_n) - p(\tilde{\lambda}_n) \right\} \right| < \frac{\epsilon}{3},\tag{105}$$

for all $N > N_0$. We use the triangle inequality for (93) to obtain

$$\begin{aligned}|J| &< \frac{1}{N} \sum_{n=0}^{N-1} |\psi(\lambda_n) - p(\lambda_n)| + \frac{1}{N} \sum_{n=0}^{N-1} |p(\tilde{\lambda}_n) - \psi(\tilde{\lambda}_n)| \\ &\quad + \left| \frac{1}{N} \sum_{n=0}^{N-1} \left\{ p(\lambda_n) - p(\tilde{\lambda}_n) \right\} \right| < \epsilon,\end{aligned}\tag{106}$$

where we have used (104) and (105).

■

B. Application of Theorem 4

For notational convenience, the subscripts of Σ_M , $\mathbf{G}_{\Delta_t, \Delta_r}$, and of $\tilde{\mathbf{G}}_{\Delta_t, \Delta_r}$ are omitted. We use Theorem 4 to prove Theorem 1. Let $\{\lambda_n \geq 0\}$ and $\{\tilde{\lambda}_n \geq 0\}$ denote the eigenvalues of the two positive semi-definite Hermitian matrices $\mathbf{A} = \mathbf{G}\Sigma\mathbf{G}^H$ and $\mathbf{B} = \tilde{\mathbf{G}}\Sigma\tilde{\mathbf{G}}^H$, respectively. the left-hand side (LHS) of (27) is equivalent to

$$\frac{1}{2\Delta_r \min\{1, \alpha\}} \left| \frac{1}{N} \sum_{n=0}^{N-1} \left\{ \log(1 + \tilde{\gamma}\lambda_n) - \log(1 + \tilde{\gamma}\tilde{\lambda}_n) \right\} \right|. \quad (107)$$

From Theorem 4, the quantity (107) tends to zero if \mathbf{A} and \mathbf{B} are asymptotically equivalent for all $\Delta_t, \Delta_r \in (0, 1/2]$, covariance matrices $\Sigma \in \mathfrak{S}_M$, and all channel instances $\mathcal{C} \in \mathfrak{C}$ in the large-system limit.

From Assumption 2 and Lemma 7, it is sufficient to prove that $(2L_t)^{-1/2}\mathbf{G}$ and $(2L_t)^{-1/2}\tilde{\mathbf{G}}$ are asymptotically equivalent. Assumption 1 implies the uniform boundedness of $\|(2L_t)^{-1/2}\mathbf{G}\|$ for fixed α . Furthermore, $\|(2L_t)^{-1/2}\tilde{\mathbf{G}}\|$ is also uniformly bounded from the first property in Property 2.

Next, we shall upper-bound the normalized Frobenius norm $\|(2L_t)^{-1/2}(\mathbf{G} - \tilde{\mathbf{G}})\|_F$. From the definition (26) of $\tilde{\mathbf{G}}$, we obtain

$$\|(2L_t)^{-1/2}(\mathbf{G} - \tilde{\mathbf{G}})\|_F^2 = J_1 + J_2 + J_3, \quad (108)$$

with

$$J_1 = \frac{\Delta_r}{2L_t L_r} \sum_{n \in \mathcal{N}} \sum_{m \notin \mathcal{M}} |g_{n,m}|^2, \quad (109)$$

$$J_2 = \frac{\Delta_r}{2L_t L_r} \sum_{n \notin \mathcal{N}} \sum_{m \in \mathcal{M}} |g_{n,m}|^2, \quad (110)$$

$$J_3 = \frac{\Delta_r}{2L_t L_r} \sum_{n \notin \mathcal{N}} \sum_{m \notin \mathcal{M}} |g_{n,m}|^2. \quad (111)$$

We first evaluate J_1 . Using (24) yields

$$\begin{aligned} J_1 &< \frac{\Delta_r}{2L_t L_r} \sum_{n=0}^{N-1} \sum_{m \notin \mathcal{M}} |g_{n,m}|^2 \\ &= 2\Delta_r \int a(p)a^*(p') I_1(\Omega_r(p), \Omega_r(p'); L_r, \Delta_r) \\ &\quad \cdot I_2(\Omega_t(p), \Omega_t(p'); L_t, \Delta_t) dp dp', \end{aligned} \quad (112)$$

with (10) and (14). From Lemmas 1 and 2, $s_L J_1$ is uniformly bounded for all $\Delta_t, \Delta_r \in (0, 1/2]$ and all channel instances $\mathcal{C} \in \mathfrak{C}$ in the large-system limit.

Repeating the same argument for J_2 , we find that $s_L J_2$ is uniformly bounded. Similarly, it is possible to prove that $s_L^2 J_3$ is uniformly bounded in the large-system limit. Combining these observations, we arrive at $\|(2L_t)^{-1/2}(\mathbf{G} - \tilde{\mathbf{G}})\|_F \rightarrow 0$ uniformly for all $\Delta_t, \Delta_r \in (0, 1/2]$ and all channel instances $\mathcal{C} \in \mathfrak{C}$ in the large-system limit.

APPENDIX D PROOF OF THEOREM 2

Theorem 2 is proved by repeating the proof of Theorem 1, while Lemma 3 is used instead of Lemma 2. Thus, we use the same notation as in the proof of Theorem 1.

For notational convenience, define the shrunk channel matrix $\tilde{\mathbf{G}}_{\Delta_t, \Delta_r} \in \mathbb{C}^{(2L_r+1) \times (2L_t+1)}$ obtained by eliminating the all-zero columns and rows from the channel matrix (26), and subsequently by moving the L_t th column and L_r th row to the last positions. Thus, we should consider the covariance matrix $\tilde{\Sigma}_{2L_t+1}$ obtained by moving the L_t th column and row of Σ_{2L_t+1} to the last positions, as well as the extended matrix $\tilde{\mathbf{G}}_{1/2, 1/2} \in \mathbb{C}^{(2L_r+1) \times (2L_t+1)}$ obtained by inserting all-zero vectors after the last column and row of the matrix $\mathbf{G}_{1/2, 1/2}$.

Let $\{\lambda_n \geq 0\}$ and $\{\tilde{\lambda}_n \geq 0\}$ denote the eigenvalues of the two Hermitian matrices $\mathbf{A} = \tilde{\mathbf{G}}_{1/2, 1/2} \mathbf{E}_{2L_t, 1} (\Sigma_{2L_t}) \tilde{\mathbf{G}}_{1/2, 1/2}^H$ and $\mathbf{B} = \tilde{\mathbf{G}}_{\Delta_t, \Delta_r} \tilde{\Sigma}_{2L_t+1} \tilde{\mathbf{G}}_{\Delta_t, \Delta_r}^H$, respectively. The LHS of (29) is equivalent to

$$\frac{1}{\min\{1, \alpha\}} \left| \frac{1}{2L_r} \sum_{n=0}^{2L_r} \left\{ \log(1 + \tilde{\gamma}\lambda_n) - \log(1 + \tilde{\gamma}\tilde{\lambda}_n) \right\} \right|. \quad (113)$$

Assumption 2 and the condition (28) imply that $(2L_t + 1)\mathbf{E}_{2L_t,1}(\mathbf{\Sigma}_{2L_t})$ and $(2L_t + 1)\bar{\mathbf{\Sigma}}_{2L_t+1}$ are asymptotically equivalent. From the proof of Theorem 1, it is sufficient to prove that $K = \|(2L_t + 1)^{-1/2}(\bar{\mathbf{G}}_{1/2,1/2} - \bar{\mathbf{G}}_{\Delta_t,\Delta_r})\|_{\text{F}}^2$ uniformly converges to zero in the large-system limit. From (24) and the definition of $\bar{\mathbf{G}}_{\Delta_t,\Delta_r} \in \mathbb{C}^{(2L_r+1) \times (2L_t+1)}$,

$$K = \frac{4L_t L_r}{(2L_t + 1)(2L_r + 1)} \cdot \int a(p)a^*(p') \sum_{n=0}^{2L_r} \sum_{m=0}^{2L_t} \tilde{D}_{n,m}(p) \tilde{D}_{n,m}^*(p') dp dp', \quad (114)$$

with

$$\begin{aligned} \tilde{D}_{n,m}(p) = & (1 - \delta_{n,2L_r})(1 - \delta_{m,2L_t})f_{1/2}(n,p)f_{1/2}(m,p) \\ & - f_{\Delta_r}(\kappa_{L_r,\Delta_r}(n),p)f_{\Delta_t}(\kappa_{L_t,\Delta_t}(m),p), \end{aligned} \quad (115)$$

with the mapping $\kappa_{L,\Delta}(k)$ given by (16). In (115), the abbreviations $f_{\Delta_r}(n,p) = f_{L_r,\Delta_r}(n/L_r - \Omega_r(p))$ and $f_{\Delta_t}(m,p) = f_{L_t,\Delta_t}(m/L_t - \Omega_t(p))$ have been introduced. Note that n and m are not dummy variables but the indices of receive and transmit antennas, respectively.

Substituting the identity

$$\begin{aligned} \tilde{D}_{n,m}(p) = & [(1 - \delta_{n,2L_r})f_{1/2}(n,p) - f_{\Delta_r}(\kappa_{L_r,\Delta_r}(n),p)] \\ & \cdot (1 - \delta_{m,2L_t})f_{1/2}(m,p) \\ & + [(1 - \delta_{m,2L_t})f_{1/2}(m,p) - f_{\Delta_t}(\kappa_{L_t,\Delta_t}(m),p)] \\ & \cdot f_{\Delta_r}(\kappa_{L_r,\Delta_r}(n),p) \end{aligned} \quad (116)$$

into (114), we have

$$K = \frac{4L_t L_r}{(2L_t + 1)(2L_r + 1)} (K_1 + K_2 + K_2^* + K_3), \quad (117)$$

with

$$\begin{aligned} K_1 = & \int a(p)a^*(p')I_3(\Omega_r(p),\Omega_r(p');L_r,\Delta_r) \\ & \cdot I_1(\Omega_t(p),\Omega_t(p');L_t,1/2)dpdp', \end{aligned} \quad (118)$$

$$\begin{aligned} K_2 = & \int a(p)a^*(p') \sum_{n=0}^{2L_r} f_{\Delta_r}^*(\kappa_{L_r,\Delta_r}(n),p') \\ & \cdot [(1 - \delta_{n,2L_r})f_{1/2}(n,p) - f_{\Delta_r}(\kappa_{L_r,\Delta_r}(n),p)] \\ & \cdot \sum_{m=0}^{2L_t-1} f_{1/2}(m,p) \\ & \cdot [f_{1/2}(m,p') - f_{\Delta_t}(\kappa_{L_t,\Delta_t}(m),p')]^* dpdp', \end{aligned} \quad (119)$$

$$\begin{aligned} K_3 = & \int a(p)a^*(p')I_3(\Omega_t(p),\Omega_t(p');L_t,\Delta_t) \\ & \cdot \sum_{n=0}^{2L_r} f_{\Delta_r}(\kappa_{L_r,\Delta_r}(n),p)f_{\Delta_r}^*(\kappa_{L_r,\Delta_r}(n),p')dpdp', \end{aligned} \quad (120)$$

where I_1 and I_3 are given by (10) and (17), respectively.

We first upper-bound K_1 and K_3 . Under Assumption 1, Lemmas 1 and 3 imply that $s_L|K_1|$ is uniformly bounded for all channel instances and antenna separations in the large-system limit. Similarly, we find the uniform boundedness of $s_L|K_3|$ from the upper bound

$$\begin{aligned} & \left| \sum_{n=0}^{2L_r} f_{\Delta_r}(\kappa_{L_r,\Delta_r}(n),p)f_{\Delta_r}^*(\kappa_{L_r,\Delta_r}(n),p') \right|^2 \\ & \leq \sum_{n=0}^{2L_r} |f_{\Delta_r}(\kappa_{L_r,\Delta_r}(n),p)|^2 \sum_{n'=0}^{2L_r} |f_{\Delta_r}(\kappa_{L_r,\Delta_r}(n'),p')|^2 \\ & < |I_1(\Omega_r(p),\Omega_r(p);L_r,\Delta_r)|^2. \end{aligned} \quad (121)$$

In the derivation of the first inequality, we have used the Cauchy-Schwarz inequality.

We next evaluate $|K_2|$. Using the Cauchy-Schwarz inequality, we have

$$|K_2| \leq \int |a(p)||a(p')| \left\{ \sum_{n=0}^{2L_r} |f_{\Delta_r}(\kappa_{L_r, \Delta_r}(n), p')|^2 \cdot I_3(\Omega_r(p), \Omega_r(p'); L_r, \Delta_r) |I_1(\Omega_t(p), \Omega_t(p); L_t, 1/2)| \cdot I_3(\Omega_t(p'), \Omega_t(p'); L_t, \Delta_t) \right\}^{1/2} dp dp'. \quad (122)$$

Upper-bounding the sum in (122) yields

$$|K_2| < \int |a(p)| \{ |I_3(\Omega_r(p), \Omega_r(p); L_r, \Delta_r)| \cdot |I_1(\Omega_t(p), \Omega_t(p); L_t, 1/2)| \}^{1/2} dp \cdot \int |a(p')| \{ |I_1(\Omega_r(p'), \Omega_r(p'); L_r, \Delta_r)| \cdot |I_3(\Omega_t(p'), \Omega_t(p'); L_t, \Delta_t)| \}^{1/2} dp'. \quad (123)$$

Thus, $s_L |K_2|$ is also uniformly bounded. Combining these observations, we arrive at Theorem 2.

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